Cardinality and Utilitarianism through social interactions
Online supplement
An example of the non-uniqueness of weights

The following example demonstrates that under assumptions A1-A6 from the paper the weights on individual utilities within the utilitarian representation of group preferences is not necessarily unique. The problem arises when individual utilities are linearly dependent. For the example we set weights to sum to one, but that is only for convenience. The same utilities could be expressed with another normalization (recall that utilities are jointly cardinal by our theorem).

Consider individuals $\{1, 2, 3\}$ and the utilities representing the preferences of all possible sub-groups. Suppose that there are outcomes x_* and x^* such that $x^* \succ^i x_*$ for $i = 1, 2, 3$, and normalize all utilities to return 0 for x_* and 1 for x^*. Suppose that $U_{\{1,2\}} = 0.5U_1 + 0.5U_2$ and $U_{\{1,3\}} = 0.5U_1 + 0.5U_3$. you may think of $\{1, 2\}$ and $\{1, 3\}$ as being organic groups, so that their preferences are extracted from their voluntary joint decisions. Assume further that U_1, U_2, and U_3 are linearly dependent, such that $U_3 = 0.8U_1 + 0.2U_2$. The utility of $\{2, 3\}$ can then be $U_{\{2,3\}} = 0.5U_2 + 0.5U_3$, as implied when considering the relative weights of $1, 2$ and $1, 3$, but may also be, for example, $U_{\{2,3\}} = 0.375U_2 + 0.625U_3$, or $U_{\{2,3\}} = 0.75U_2 + 0.25U_3$. In the latter two cases, it will be impossible for $U_{\{1,2,3\}}$ to maintain the same relative weights between individual utilities as in all three utilities $U_{\{1,2\}}, U_{\{1,3\}}$, and $U_{\{2,3\}}$, even though $U_{\{1,2,3\}}$ can be set to satisfy (ii) of Theorem 1 in the paper.[1]

Although when individual utilities are linearly dependent there is some freedom in setting weights within utilitarian sums representing group preferences, Theorem 1 in the paper still imposes restrictions on the choice of weights. For instance, if $U_{\{2,3\}} = 0.25U_2 + 0.75U_3$, then $U_{\{1,2,3\}}$ cannot be set to $0.9U_{\{1,2\}} + 0.1U_3 = 0.45U_1 + 0.45U_2 + 0.1U_3$, because in that case $U_{\{1,2,3\}}$ cannot be expressed as a convex combination of U_1 and $U_{\{2,3\}}$. The choice of weights therefore becomes more complicated. When imposing the additional

[1] For instance, $U_{\{1,2,3\}} = \frac{2}{3}U_{\{1,2\}} + \frac{1}{3}U_3 = \frac{2}{3}U_{\{1,3\}} + \frac{1}{3}U_2 = \frac{1}{3}U_1 + \frac{1}{3}U_2 + \frac{1}{3}U_3 = 0.6U_1 + 0.4U_2$ can be represented as a convex combination of $U_{\{2,3\}}$ and U_1 in both above choices.

assumptions required for Theorem 2 in the paper, we obtain $U_{\{2,3\}} = 0.5U_2 + 0.5U_3$ and $U_{\{1,2,3\}} = \frac{1}{3}U_1 + \frac{1}{3}U_2 + \frac{1}{3}U_3$.