Competitive equilibrium as a bargaining solution: an axiomatic approach*

Shiri Alon† and Ehud Lehrer‡

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Abstract

The paper introduces an axiomatic characterization of a solution to bargaining problems. Bargaining problems are specified by: (a) the preference relations of the bargaining parties (b) resources that are the subject of bargaining, and (c) a pre-specified disagreement bundle for each party that would result if bargaining fails. The approach is ordinal in that parties' preferences are over bundles of goods and do not imply any risk attitudes. The resulting solution is accordingly independent of the specific utilities chosen to represent parties' preferences. We propose axioms that characterize a solution matching each bargaining problem with an exchange economy, and assigning the set of equilibrium allocations corresponding to one equilibrium price vector of that economy. The axioms describe a solution that results from an impartial arbitration process, expressing the view that arbitration is a natural method to settle disputes in which agents have conflicting interests, but can all gain from compromise.

Keywords: Bargaining, exchange economy, ordinal bargaining solution, competitive equilibrium.
JEL classification: C70; C78; D51; D58

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†Bar-Ilan University, Ramat-Gan 5290002, Israel. e-mail: Shiri.Alon-Eron@biu.ac.il

‡The School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel and INSEAD, Boulevard de Constance, 77305 Fontainebleau, France. e-mail: lehrer@post.tau.ac.il

1 Introduction

In many business disputes the source of controversy between the conflicting parties concerns the distribution of assets among them. One of the most common methods for dispute resolution, alternative to litigation, is arbitration. When the conflicting parties choose to undergo arbitration they often reach quicker, more efficient, and sometimes less costly resolutions, compared to those they would have reached through conventional court proceedings. In contrast to litigation, which can be initiated whenever one of the conflicting parties decides to resort to court, a key feature of arbitration is that it can be initiated only upon the unanimous consent of the parties involved, and conducted according to principles they all agree on.

When a dispute over assets arises each party has a default allocation that can be achieved even when no agreement is reached. Surely a default of no assets can always be achieved, but in some cases, when a clear ownership of assets is involved, for instance when one of the disputing parties owns a real estate property, a more sophisticated disagreement result obtains. Nonetheless, in case of disagreement parties usually end up in a less favorable position, whereby the prolongation of the conflict delays the distribution of assets (or at least of those assets that remain after the parties attain their defaults). This could be the case, for instance, if the dispute ends up in a lengthy court litigation.

The current paper addresses disputes over multiple assets, involving several parties. Disputes are described by a bundle of assets that should be allocated to a number of parties, and by the preferences of those parties over assets (without supposing that there is any risk involved). Preferences are only assumed to be ordinal, namely there is no notion of strength of preference and the only significant question when comparing two bundles is which of them is preferred over the other. It is supposed that each concerned party can prevent unilaterally an agreement (e.g. turn to court), in which case each of the involved parties will obtain a pre-defined default bundle. The problems handled are non-trivial in the sense that all parties maintain that the dispute may be resolved in more beneficial ways compared to their default results, hence all parties involved have an incentive to compromise. On the other hand, each party retains the option to influence the obtained compromise by imposing unilaterally the inferior default result. A solution is sought, that would assign to each dispute allocations of the assets under consideration.

Given that each entity involved in a dispute can cause the breakdown of an agreement, a solution would provide a viable method of dispute resolution only if parties are persuaded to accept it. Finding such a solution is our goal in the current essay, which offers an axiomatic characterization of dispute resolution.

When entering a business relationship, parties often sign a contractual agreement that requires them to resolve possible future disputes by means of an arbitration process. The contract furthermore specifies the principles by which arbitration will take place. Analogously to such a contract, our axioms formulate arbitration principles, describing a solution that results from an impartial arbitration process. We then show that a solution satisfying these axioms must be of a specific functional form (and vice versa). Namely, agreeing to an arbitration process that accords with the proposed principles is equivalent to employing the functional mechanism.

The problems we address consist of allocating goods to agents. The primary economic mechanism that is designed for that task is an exchange economy. Given initial bundles of goods, one bundle per each agent, a competitive equilibrium of the economy determines which allocation prevails: what goods each agent sells and how much each agent decides to consume. The solution concept characterized in this paper, termed *the market bargaining solution*, utilizes the market mechanism as an instrument. This is done by associating each dispute with an exchange economy, selecting an equilibrium price vector for that economy and assigning as a resolution all the equilibrium allocations that correspond to the selected equilibrium price vector (the solution therefore depends on the selection and is not unique). The exchange economy that is matched to a dispute is the one involving agents with the preferences of the disputing parties, each endowed with the corresponding party's default allocation plus an equal share of the remainder. Each concerned party is therefore considered to be in possession of his or her default result, with the remaining assets divided equally. Our axiomatic characterization establishes that equilibrium allocations can be viewed as resolutions to a dispute rendered by an impartial arbitrator. An interesting feature of the development is that equilibrium prices are derived endogenously per dispute, and interpreted as representing an impartial arbitrator, who evaluates bundles at their price worth.

The problems handled in this paper are similar to classic bargaining problems as formulated by Nash [14]. Both types of problems concern the distribution of resources among a group of agents. In both cases a solution determines, for each problem, how resources are to be allocated to the parties involved, where each party is assumed to be able to reject compromise and unilaterally impose an inferior result on all parties involved. The axiomatic approach, aimed at attaining a consensus among disputing parties, is prevalent in the bargaining literature. The difference between the two types of problems lies in the assumptions made on the parties' preferences over resources, and in the formulation of feasible allocations and disagreement results.

Under the classic bargaining approach problems are formulated in terms of utilities. The default result, as well as feasible allocations of resources, are described in utility terms, suppressing any reference to the actual assets that underlie those utilities. Consequently, any two distinct economic environments that generate the same utility image will inevitably be assigned the same allocation as a solution. The classic bargaining approach relies on the assumption that meaningful information regarding the cardinality of parties' preferences is available, for instance due to preferences being defined over lotteries. However, for some problems such information is not attainable. Roemer, [18] and [19], discusses the implications of assuming that bargaining problems can be formulated in terms of utilities, and suggests a more general bargaining setup that is defined in terms of resources and preferences. This is the setup we employ in this paper, and accordingly, the solution we offer is specified in terms of assets. The difference between the resource-based bargaining approach and that of the classic bargaining literature can be illustrated through two simple examples.

Imagine two business partners engaged in a dispute over one acre of land, where in case of disagreement both get nothing. Each of the partners naturally wants to have as much of the land as possible. If no cardinal information regarding the partners' preferences is available, then their preferences for more land can be represented by any strictly increasing function of x, x being the received fraction of the land. Consider two such representations, $u_1(x) = u_2(x) = x$ on one hand, and $u_1(x) = \sqrt{x}$ and $u_2(x) = 1 - \sqrt{1-x}$ on the other, where u_i stands for partner i's utility from the land. Both representations induce the same utility-possibility set, namely the same set of utility allocations (these are the points in the triangle bounded within the origin and the points $(0,1)$ and $(1,0)$), and the same $(0,0)$ default result in case of disagreement. Hence, every bargaining solution concept which employs the classic bargaining setup will match both problems with the same solution, that is with the same allocation of utilities. Symmetry of the utility-possibility set would lead to a fifty-fifty distribution of utilities under all well-known bargaining solution concepts. However, while under the first utilities representation this solution translates to assigning each of the partners half the acre of land, in the second utilities representation it entails allocating $1/4$ of the land to one partner and $3/4$ to the other. By contrast, a market bargaining solution will entail an equal division of the land regardless of the choice of specific utility representations, since underlying preferences over assets and the assets available for division remain unchanged.

Aside from assuming only ordinality of preferences, another difference between the two approaches is manifested in the dependence of a resolution on parties' disagreement results. To demonstrate this difference, suppose two partners with identical Cobb-Douglas preferences disputing over one acre of land and one ton of wheat seed. Imagine that in case bargaining fails the first partner gets all the land while her or his rival gets nothing (let's say, because the first partner owned the land before entering the partnership). Any utility-based solution will translate those disagreement bundles into a symmetric disagreement point, as utilities are both zero. Yet it is doubtful that the partner owning the land will concede to such a formulation of the dispute, as this would mean ignoring differences related to investment size and initial ownership of assets. Our approach accounts for such differences by expressing the problem, including parties' default results, in terms of assets.

On the negative side, an inevitable limitation of the suggested approach is that it applies only to bargaining problems whose corresponding exchange economies possess competitive equilibria. To that end, the paper characterizes a solution under assumptions that guarantee the existence of such an equilibrium. As is the case in market-related questions in general, here as well multiple equilibrium price vectors may exist. A market bargaining solution selects one of them, subject to one minor consistency requirement across different problems. Admittedly, parties may not be indifferent to this selection, as their welfare under one equilibrium price vector may be higher than under another. A comparison between different selection schemes might be of interest, but it lies beyond the scope of this paper.

1.1 Literature review

The current paper touches on three central topics in economics: Competitive equilibrium, in the solution outlined; Bargaining Theory, in considering bargaining problems; And social choice theory, in the special case of parties with equal starting positions. As there is an abundance of papers studying these topics and the relationships between them, we confine ourselves to discussing only the most relevant ones.

A framework which is analogue to ours is that of an open economy, where agents trade initial endowments, as well as an additional exogenous bundle of goods. Korthues [9] investigates allocation solutions in such economies, and characterizes equilibrium allocations in a corresponding standard economy, where agents' budget constraints are proportional to the constraints arising from their initial endowments. The characterization in Korthues relies on a consistency condition, involving an auxiliary economy that is built of all possible permutations of commodities, and on a proportionality condition for economies of a specific type.

In the special case where parties enter a dispute with the same disagreement bundles, the solution offered here is competitive equilibrium from equal division of resources. This is a prominent solution concept in the theory of fair allocations, and is known to satisfy many socially desirable attributes such as no envy, individual rationality, efficiency, and various consistency conditions. Competitive equilibrium from equal division was characterized axiomatically in various papers, for example Thomson, [26] and [27], Nagahisa, [12] and [13], and Nagahisa and Suh [11]. These axiomatizations can be 'plugged in' as a solution to disputes in which the disagreement point is symmetric. However, the general case, in which parties possess possibly disparate starting positions, is not accommodated by those papers, as naturally parties within the social choice literature are treated equally. Hence, these papers do not characterize a resolution of disputes over assets, as we present here.

This paper handles bargaining problems that are formulated in terms of assets and ordinal preferences, rather than in terms of utility values. Roemer, in [18] and [19], argues for the need of this kind of formulation, which allows for a richer variety of solution concepts. Roemer characterizes classic bargaining solution in this richer setup and demonstrates the strength of assumptions required for that characterization. Sertel and Yildiz [24] show that within the classic bargaining setup it is impossible to define a solution which for every utility image of an exchange economy assigns utilities of respective Walrasian allocations.

There are several other solutions that are characterized within an ordinal setup. Chen and Maskin [3] extend the setup to include production and characterize an egalitarian solution, equating agents' utilities, under a requirement that agents' utilities do not decrease following an improvement in technology. Perez-Castrillo and Wettstein [17] employ a condition involving contributions of agents to sub-groups, generalizing attributes of the Shapley value. They characterize a solution which generalizes the Pareto efficient egalitarian-equivalence (PEEE) solution of Pazner and Schmeidler [16] to the case of non-equal initial positions. PEEE allocations, designed as a fair solution to resource allocation problems, form a resolution to disputes that are based on ordinal preferences, as in the setup employed here, as long as the disagreement bundles are the same for all agents. These are allocations which for each agent are equivalent preference-wise to the same fixed bundle. Thus, they are preference-wise equivalent to egalitarian allocations in some hypothetical economy. A disadvantage of the PEEE solution from the perspective of disputing agents is that it may allocate to one agent a bundle which dominates, good-by-good, another agent's bundle (when there are more than two parties to a dispute). In the course of a dispute, such a finding may lead to a denial of compromise.

Another characterization of a bargaining solution in an ordinal setup is Nicolo and Perea [15]. Their paper addresses two-persons bargaining problems, also formulated by means of resources and preferences (under some conditions their development can be applied to problems with any number of agents). Their solution depends on an exogenously given increasing family of allocation sets, and their characterization hinges on a form of monotonicity with respect to enlargement of possibility sets, and on a condition that binds together problems with different preferences.

In the problems that we have in mind agents agree in advance, prior to engaging in a joint business and before any dispute arises, to employ an arbitration mechanism as depicted in the axioms. It follows that the parties to be involved in a dispute are fixed. Thus, assumptions that involve possible changes concerning the nature of the involved agents (e.g., assumptions referring to sub-groups or to transformed preferences) will be deemed less relevant by agents.

Among the papers that take a preference-based approach to bargaining is Rubinstein, Safra and Thomson [21], that describes two-persons bargaining problems in terms of preferences over lotteries. In that paper a solution is defined by means of preferences, and is shown to coincide with the famous Nash bargaining solution [14] whenever preferences admit an expected utility representation. The result of Rubinstein, Safra and Thomson is extended in Grant and Kajii [6] to additional preference types. Essentially, the preferences and solutions considered in these works are cardinal. That is to say, preferences are represented by a utility function that is invariant only under positive affine transformations (where this type of uniqueness is the result of using a setup that contains lotteries), hence the solution to a bargaining problem may change following different order-preserving representations of preferences over assets. This differs from our framework, which addresses preferences over assets which are ordinal in nature.

Our ordinal approach should not be confused with ordinal solutions to utility-formulated bargaining problems. The latter approach, as in Shapley [25] and Safra and Samet [22], among others, still considers classic bargaining problems that are described by means of agents' utility levels. Therefore, it cannot distinguish between different disputes that induce the same utilities image (as in the examples above).

Other types of links between bargaining and competitive equilibria are explored in Trockel [28] and in Dávila and Eeckhout [4]. Trockel, employing an economy as a solution device, considers bargaining problems described in terms of utilities (not in terms of preference orders), and matches each one with an Arrow-Debreu economy the (unique) competitive equilibrium of which identifies with a corresponding asymmetric Nash bargaining solution. Dávila and Eeckhout [4] describe a bargaining procedure between two agents, involving offers on price and maximum quantity to be traded. They show that when agents become infinitely patient the bargaining result converges to a competitive equilibrium allocation (thus depending only on agents' preferences and endowments, rather than on bargaining-related characteristics).

Finally we mention a few well-known works in Bargaining Theory. Nash [14] was the first to formulate what is now known as the Nash Bargaining Problem: a two-persons setup in which parties can collaborate in a way that will be beneficial for both, and need to agree on the utility values that each will gain from this collaboration, otherwise they will obtain inferior disagreement values. Nash [14] phrases the problem in terms of utilities and offers an axiomatic treatment under the assumption that utility is cardinal (unique up to positive affine transformations). The axioms are shown to lead to a unique solution in terms of utilities. Later works suggest alternative axiomatizations and solutions to the Nash Bargaining Problem, still considering a utility-based formulation. Among those are Kalai and Somorodinsky [8], Kalai [7], and many others.

2 Characterization and result

2.1 Setup and notation

Suppose a set of divisible goods, $\{1, \ldots, L\}$, and consider bundles of these goods which are elements in $\mathbb{R}_+^L$. For two bundles $x, y \in \mathbb{R}_+^L$ we write $x \geq y$ whenever $x_\ell \geq y_\ell$ for every $\ell = 1, \ldots, L$, and $x >> y$ whenever $x_\ell > y_\ell$ for every $\ell = 1, \ldots, L$. The set of allocations of bundles to a group of n agents is $\mathcal{A}_n = (\mathbb{R}_+^L)^n$. An element of $\mathcal{A}_n$, termed an *allocation*, is denoted by $a = (a(1), \ldots, a(n))$, where $a(i) = (a_1(i), \ldots, a_L(i))$ is the bundle allocated to the i-th agent, containing quantity $a_\ell(i)$ of good ℓ. For given resources $x = (x_1, \ldots, x_L)$, $\mathcal{A}_n(x)$ denotes the subset of $\mathcal{A}_n$ consisting of distributions of x to n parties, namely of allocations a that satisfy, $\displaystyle\sum_{i=1}^{n} a_\ell(i) = x_\ell$ for every good ℓ.

The bargaining problems addressed are triplets, $(x, (\succsim^i)_{i=1}^n, d)$, consisting of resources $x \in \mathbb{R}_+^L$ that need to be split between agents $i = 1, \ldots, n$, endowed with preferences over bundles given by binary relations over $\mathbb{R}_+^L$, $(\succsim^i)_{i=1}^n$, where the disagreement point is $d = (d(1), \ldots, d(n)) \in \mathbb{R}_+^L$, $d(i)$ being the bundle to be allocated to party i in case bargaining fails. The asymmetric and symmetric parts of each $\succsim^i$ are respectively denoted by $\succ^i$ and $\sim^i$. As explained in the Introduction, in order for the market bargaining solution to be well defined it is essential that a competitive equilibrium exists in any problem considered. To guarantee this, a structural assumption is employed.

A0. Structural assumption.
Any bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$ satisfies the following assumptions:

(1) $x >> \displaystyle\sum_{i=1}^{n} d(i)$

(2) For every i and $\succsim^i$ over $\mathbb{R}_+^L$ it holds that:

  (a) $\succsim^i$ is a weak order, namely, it is complete and transitive.

  (b) $\succsim^i$ is monotonic: for any $y, z \in \mathbb{R}_+^L$, if $y >> z$ then $y \succ^i z$.

  (c) $\succsim^i$ is continuous: for every $y \in \mathbb{R}_+^L$ the sets $\{z \in \mathbb{R}_+^L \mid z \succsim^i y\}$ and $\{z \in \mathbb{R}_+^L \mid y \succsim^i z\}$ are closed.

(d) $\succsim^i$ is convex: for every $y, z \in \mathbb{R}_+^L$, if $y \succ^i z$ then $\lambda y + (1-\lambda)z \succ^i z$ for every $\lambda \in (0,1)$.

The conditions described in (2)(a) through (2)(d) are the standard Arrow-Debreu [1] conditions. They do, however, limit the types of disputes that we can address. For instance, disputes in which agents are interested only in part of the available resources, so that their preferences are satiated, are ruled out by our structural assumption.

The assumption in (1), combined with monotonicity (**A0**(2b)), implies that problems are non-trivial, in the sense that some allocations are deemed by everybody as better than the default. Technically speaking, this additional assumption guarantees that an equilibrium exists for the economy that corresponds to any bargaining problem.

A *bargaining solution* is a correspondence φ that assigns to every bargaining problem that satisfies the structural assumption above, a nonempty set of distributions of x to the agents $i = 1, \ldots, n$. That is, $\varphi(x, (\succsim^i)_{i=1}^n, d)$ is a subset of $\mathcal{A}_n(x)$.

2.2 Assumptions and main result

Three attributes are assumed on a bargaining solution φ. The first assumption builds on the fact that agents in our framework are fully described by their preferences, and each agent is supposed to be concerned only with his or her own wellbeing. Thus, if two allocations yield for an agent bundles between which the agent is indifferent, then those two allocations will be regarded by the agent as equivalent. The second assumption maintains that all the agents deem all the allocations assigned by a given solution as equivalent in that sense. The solution therefore satisfies the property of *single-valuedness*, in the terminology of Moulin and Thomson [10]. Namely, the solution is, preference-wise, a singleton. If this were not the case, agents could not be expected to agree on any particular allocation assigned by a solution, since each allocation assigned could yield them a different utility level. The axiom further states that if two allocations are equivalent in any relevant respect (i.e., considered equivalent by any of the agents), either both of them are assigned by the solution or none of them is. A solution is therefore a *full correspondence*, as defined in Roemer [19].

A1. Equivalence Principle.

Suppose a bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$ and two allocations, $a, a' \in \mathcal{A}_n(x)$. If $a, a' \in \varphi(x, (\succsim^i)_{i=1}^n, d)$ then $a(i) \sim^i a'(i)$ for every i. In the other direction, if, for every i, $a(i) \sim^i a'(i)$, then $a \in \varphi(x, (\succsim^i)_{i=1}^n, d)$ if and only if $a' \in \varphi(x, (\succsim^i)_{i=1}^n, d)$.

A bargaining situation demands discretional agreement of all the agents, otherwise the disagreement point, unfavorable to all agents, will prevail. The need for unanimous consent among agents creates tension between their incentive to compromise and the power held by each agent to threaten the others with the breakdown of negotiations. An involvement of a third party is often required, and welcomed by agents, as a way to resolve this tension. A prevalent dispute resolution method, administered by a third party, is arbitration. In many business partnerships parties agree in advance that any future dispute among them will be resolved by arbitration, and specify beforehand its fundamentals.

Our second axiom requires that any allocation assigned by a solution be the result of arbitration conducted according to two basic principles. Hence, agents subscribing to this axiom accept this type of arbitration as a dispute resolution method. In conformity with the representation of parties to a dispute by their preferences, an arbitrator in our framework is modelled by a preference relation over bundles of goods. Any ranking of bundles by such an arbitrating preference relation should be understood as reflecting the principles that guide the decisions of an arbitrator rather than an expression of this arbitrator's personal preferences. Importantly, although an arbitrating preference is defined over bundles of goods, the two axioms that apply to such a relation treat all individuals' bundles in a symmetric manner. Namely, the two axioms impose the same conditions on each bundle within an allocation, thus yielding that the conditions involved pertain to allocations as a whole and not to single bundles.

The first principle expressed in our second axiom is that the arbitrator resolving a bargaining problem should be impartial. Two conditions are imposed to maintain impartiality of the arbitrator. In the first it is required that arbitration be unprejudiced, in that no possible resolution be eliminated a-priori. The second condition states that suggested resolutions should be fair in the sense that after taking into account different starting positions, the arbitrator should attempt to equitably distribute the remaining resources. Each of these impartiality conditions is conveyed in a definition. In the first we identify a preference relation as unprejudiced whenever no allocation of resources to a number of parties is dominated by another allocation of the same resources to the same number of parties.

Definition 1. A binary relation $\succsim^*$ is *unprejudiced* if for every $y \in \mathbb{R}^L_+$ and $k \in \mathbb{N}$, and every $a, b \in \mathcal{A}_k(y)$, it cannot be the case that $a(i) \succ^* b(i)$ for every $i = 1, \ldots, k$.

When a preference relation reflects the decisions of an arbitrator, having one allocation dominated by another entails that the dominated allocation will never be assigned by the arbitrator. A dominated allocation will therefore be censored by the arbitrator regardless of the specifics of a bargaining problem. Allocations, however, should be eliminated or chosen only based on the preferences and starting positions of disputing parties, otherwise solutions may be renounced even if they may turn out to be favorable to the parties involved.

To understand the second impartiality definition, stating what it means for an allocation to be considered fair in our setup, recall that parties may start off with different fallback options. For instance, one agent may be in possession of some of the assets while others are not. In such a case this agent will naturally expect her or his ownership of assets to be taken into account in the arbitral decision, perceiving settlements of the dispute which ignore different a-priori rights as unfair. Our definition of fairness thus depends on the disagreement point. It states that a preference relation perceives an allocation as fair given a disagreement point whenever the surplus generated for any of the parties, on top of each party's default result, is deemed equivalent by the arbitrator.

Definition 2. A binary relation $\succsim^*$ *perceives an allocation* $a \in \mathcal{A}_n$ *as fair given a disagreement point* d if there exist bundles $t_i \in \mathbb{R}_+^L, i = 1, \ldots, n$, such that for every i, $a(i) \sim^* d(i) + t_i$, and for every i and j, $t_i \sim^* t_j$.

Definition 2 allows us to express the most basic standard of arbitration, namely an impartial treatment of parties. In the special case where agents' claims are equal, agents will expect an impartial arbitrator to rule fairly and assign allocations which the arbitrator deems equitable. Stated differently, agents will not respect the authority of an arbitrator who intentionally assigns better bundles to their peers. In the case of equal starting positions the definition above will translate, for an unprejudiced arbitrating preference, to requiring $a(i) \sim^* a(j)$ for every i and j. Generalizing this intuition, when different agents have different starting positions, these will have to be respected by the arbitrator. However, once starting positions are taken into account, any addition on top of them will be equitable.

An arbitrator in our model, attempting to resolve a dispute over resources, is assumed to satisfy the two notions of impartiality described above. But furthermore, within those limits of impartiality, an arbitrator is supposed to comply with agents' preferences. Two notions of compliance with agents' preferences are employed. First and more general, an arbitrating preference is required to abide by the preference principles that are agreed upon by all disputing agents, as specified in A0(2). For instance, as any of the agents prefers mixtures of two bundles to the less preferred bundle of the two, the arbitrating preference follows the same pattern of preference (convexity). The second, more specific notion of compliance with agents' preferences, dictates that of all those allocations that the arbitrator considers to be a fair resolution of the dispute, the ones chosen are the best, according to all agents (note that it is not a-priori clear that there exists such a Pareto-dominating allocation within the set of fair allocations; Our theorem, however, guarantees existence). Each agent can therefore trust that after applying fairness considerations, the arbitrator will allocate her or him the best possible bundle.

The next axiom, Impartial Arbitration, describes a solution to a bargaining problem as the result of an impartial arbitration process, which further complies with disputing agents' preferences. It does so by stating that any suggested resolution of the dispute admits an arbitrating preference that: (a) is unprejudiced, (b) judges the resolution as fair given agents' starting positions, (c) abides by the same preference principles that are agreed upon by all disputing agents, and (d) selects, within the domain of fair resolutions, only those which are best in all agents' eyes.

When disputing parties adopt Impartial Arbitration they commit to resolving disputes by means of an arbitration conducted in line with the principles listed above. The parties agree on an arbitrator chosen from among candidates who approach the problem open-mindedly.[1] They expect that the arbitrator be just and treat all parties equitably, once their different initial rights are taken into account. Finally, parties trust the arbitrator to respect their preferences and take those into account in the resolution suggested.

The constraint on the solution, which is to optimize only within the set of fair allocations, reflects the compromise that agents make. Agents would naturally prefer a resolution that grants them with the optimal bundle without any constraints. However, knowing that their interests conflict with those of their peers, and still they can gain from compromising, agents in our model submit to the idea of optimizing within a constrained set, constructed in an impartial manner. Compromising pays in the sense that agents ultimately end up with bundles that are strictly better for them than their default ones.

A2. Impartial Arbitration.
Let $(x, (\succsim^i)_{i=1}^n, d)$ be a bargaining problem and $a \in \varphi(x, (\succsim^i)_{i=1}^n, d)$ an allocation in its solution. Then there is a binary relation $\succsim^*$ over $\mathbb{R}_+^L$, satisfying:

(a) $\succsim^*$ is unprejudiced

(b) $\succsim^*$ perceives a as fair given d

(c) $\succsim^*$ satisfies the attributes in A0(2), namely, it is a monotone, continuous, and convex weak order

[1] In real-life disputes, arbitrators are frequently chosen from among members of a specialized institution, such as the American Arbitration Association (AAA), so that parties can trust their impartiality. As stated on the AAA website, "These neutrals are bound by AAA established standards of behavior and ethics to be fair and unbiased".

(d) For any other allocation $a' \in \mathcal{A}_n(x)$ which $\succsim^*$ perceives as fair given d, $a(i) \succsim^i a'(i)$ for every $i = 1, \ldots, n$

We then say that $\succsim^*$ arbitrates a in the problem $(x, (\succsim^i)_{i=1}^n, d)$.

Note that we do not require that an arbitrator negotiating a solution to a problem be unique. Nevertheless, our theorem will ultimately imply such uniqueness. Although there may be multiple adequate arbitrators per problem, the characterized solution selects one arbitrator, and all allocations returned by the solution are arbitrated by this one arbitrator.

Our last axiom imposes a certain form of decomposability of the solution, for bargaining problems that can be seen as composed of k identical instances of smaller problems. These are problems in which, for each preference relation involved, there are k individuals holding that preference, each with the same default allocation. The group of disputing agents can therefore be divided into k sub-groups that are identical in terms of preferences and starting positions. The axiom stipulates that a solution to the larger problem can be reached by solving one of the k replicas composing it, in which there is one instance of each preference and corresponding default allocation, and one k-th of the resources to divide. If an allocation is returned as a solution to a replica, then a replication of that allocation forms a solution to the larger problem, and moreover, both solutions, to a replica and to the large problem, can be arbitrated by the same preference. Hence specifically for bargaining problems composed of replications, the problem of finding a solution can be simplified to finding a solution to a sub-problem. This condition is similar to an attribute by the same name that was introduced by Thomson (see the survey [27] and the references therein).

A3. Replication Invariance.
Let $(kx, (\underbrace{(\succsim^i)_{i=1}^n, \ldots, (\succsim^i)_{i=1}^n}_{k \text{ times}}), (\underbrace{d, \ldots, d}_{k \text{ times}}))$ be a bargaining problem, for $k \in \mathbb{N}$.
If $a \in \varphi(x, (\succsim^i)_{i=1}^n, d)$, and $\succsim^*$ arbitrates it in $(x, (\succsim^i)_{i=1}^n, d)$, then
$a^k = (\underbrace{a, \ldots, a}_{k \text{ times}}) \in \varphi(kx, (\underbrace{(\succsim^i)_{i=1}^n, \ldots, (\succsim^i)_{i=1}^n}_{k \text{ times}}), (\underbrace{d, \ldots, d}_{k \text{ times}}))$, and $\succsim^*$ arbitrates it in
$(kx, (\underbrace{(\succsim^i)_{i=1}^n, \ldots, (\succsim^i)_{i=1}^n}_{k \text{ times}}), (\underbrace{d, \ldots, d}_{k \text{ times}}))$.

Note that $\succsim^*$ being unprejudiced, and satisfying A0(2), does not depend on the specific problem (replicated or not), and that whenever an allocation a is perceived as fair given d by a relation $\succsim^*$, then its replication a^k is perceived as fair given $(\underbrace{d, \ldots, d}_{k \text{ times}})$ by $\succsim^*$. The effective requirement within Replication Invariance is therefore that a^k belong to the solution to the replicated problem, and that for any other allocation a' in the replicated problem that is perceived as fair by $\succsim^*$ given the default bundles, a^k is (weakly) preferred to a' by all individuals.

Our main theorem shows that under the structural assumption (**A0**), axioms **A1-A3** characterize a market bargaining solution. The primary assumption in the characterization of a solution is Impartial Arbitration (**A2**), which describes a solution as the result of an impartial arbitration process, that takes under advisement parties' preferences. Our theorem states that parties' consent to the arbitration principles encapsulated in the above axioms is equivalent to allotting each party with her or his default bundle plus an equal share of the remaining resources, and allowing parties to trade among themselves.

Note that the dispute resolution that we offer is not unique, as multiple equilibrium price vectors may exist per dispute. The axiomatization does not in general mandate that any specific equilibrium price vector of the induced exchange economy be selected. It only dictates that the selection is consistent under replications, as defined next.

Definition 3. A bargaining solution φ is *replication-consistent* if the allocations it assigns, given any bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$ and its replication $(kx, (\underbrace{(\succsim^i)_{i=1}^n, \ldots, (\succsim^i)_{i=1}^n}_{k \text{ times}}), (\underbrace{d, \ldots, d}_{k \text{ times}}))$ $(k \in \mathbb{N})$, are equilibrium allocations which correspond to the same vector of equilibrium prices.

The solution characterized in the theorem matches each dispute with an exchange economy and selects one equilibrium price vector in that economy. Following Arrow and Debreu [1], an equilibrium price vector indeed exists. The solution then assigns as a resolution of the dispute all the equilibrium allocations that are distributions of x, corresponding to the selected price vector.[2] The linear preference defined by the selected price vector is in fact the arbitrator delivering the solution in the bargaining problem. It is unprejudiced and satisfies A0(2) since it is linear. Furthermore, it evaluates each bundle by its worth according to the selected prices, thus it perceives any allocation that exhausts the budget constraint for every agent as fair given agents' disagreement bundles.

Theorem 1. *Let φ be a bargaining solution to problems $(x, (\succsim^i)_{i=1}^n, d)$ that satisfy the structural assumption **A0**. The following two statements are equivalent:*

*(i) φ satisfies assumptions **A1-A3**.*

*(ii) For any bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$ that satisfies **A0**, φ selects an equilibrium price vector in the pure exchange economy in which each agent $i \in \{1, \ldots, n\}$ holds*

[2] That is, in case of a good with a zero price the solution will assign a subset of the corresponding equilibrium allocations, composed of those equilibrium allocations that do not generate excess supply.

preferences $\succsim^i$ and is endowed with $d(i) + \frac{1}{n}\left(x - \sum_{j=1}^{n} d(j)\right)$. φ then returns the set of all equilibrium allocations distributing x that correspond to the selected equilibrium price vector. In addition, φ is replication-consistent.

Agents' different disagreement bundles represent their different starting positions in the bargaining situation. Starting position may reflect different ownership or other rights over the disputed resources, hence agents are likely to expect these starting positions to influence the bargaining resolution. Our axioms characterize a solution that takes different starting positions into account by translating them into endowments in the respective exchange economy. The endowment of an agent in that (imaginary) economy becomes his or her disagreement bundle plus an equal share of the remainder.

The endowments considered are akin in spirit to the 'Contested Garment Principle' of Aumann and Maschler [2], prescribing equal division of the contested amount, if each party i is taken to claim $d_i + (x - \sum_{j=1}^{n} d_j)$ of the resources under consideration. Such claims reflect the supposition that disputing parties accept their peers' rights over their default bundles and thus claim only what remains of the resources after deducing others' disagreement bundles. An alternative, asymmetric approach, would be to consider unequal divisions of the contested amount (e.g., proportional to the disagreement result), in the spirit of the asymmetric Nash bargaining solution or the Proportional Solution of Kalai [7]. However, we do not pursue this asymmetric extension in the current paper.

The theorem implies that one impartial arbitrator is selected per every bargaining problem. This is a linear preference relation whose vector of coefficients is an equilibrium price vector in the corresponding exchange economy. It may be that there exist multiple such linear relations that can arbitrate a resolution in a given bargaining problem. All these possible relations are unprejudiced and assign efficient allocations.

It is worth noting that the solution may be viewed as a two-step procedure in utilities space. In the first only fairness considerations, while in the second individuals' preferences are taken into account. Given a selected equilibrium price vector p, every allocation $a \in \mathcal{A}_n$ can be mapped to a point $(p \cdot a(1), \ldots, p \cdot a(n)) \in \mathbb{R}^n_+$, representing its worth in equilibrium prices (or, equivalently, its worth in arbitrator utils). The first step then consists of finding a feasible allocation of $p \cdot x$ to the disputing individuals, each wanting as much of this worth as possible. In case of disagreement individuals end up with $(p \cdot d(1), \ldots, p \cdot d(n))$. In other words, it is first required to solve a bargaining problem in $\mathbb{R}^n$, where the utility-possibility set is the utility image of $\mathcal{A}_n(y)$, $y \leq x$, under the linear utility of the selected arbitrator, with the disagreement point being $(p\cdot d(1),\ldots,p\cdot d(n))$. The solution ascribed to this first step problem is the egalitarian solution, dividing the worth over the disagreement point equally among all individuals (that is, it assigns the Proportional Solution with equal proportions, see Kalai [7]).

The second stage starts with identifying all the possible utility levels of individuals which correspond to the egalitarian solution obtained in the first step. These are utility levels $(u_1(a(1)),\ldots,u_n(a(n)))$, for every $a\in\mathcal{A}_n(x)$ that satisfies $p\cdot a(i)=p\cdot d(i)+p\cdot\frac{1}{n}\left(x-\sum_{j=1}^{n}d(j)\right)$ for every i and for some (ordinal) choice of $u_i, i=1,\ldots,n$. From among these, all the Pareto-dominating allocations (which indeed exist, per our structural assumption,) are returned as a solution.

2.3 Examples

Imagine two business partners who need to decide on the distribution of one acre of land, one million dollars and two tons of wheat seed. The resources to be split are $(1,1,2)$, where the coordinates denote acres of land, millions of dollars and tons of wheat seed, respectively. The preference relations of the two partners involved are denoted by $\succsim^1$ and $\succsim^2$. For the sake of the illustration suppose that these preferences belong to the Cobb-Douglas class with parameters $(1/2,1/3,1/6)$ for $\succsim^1$ and parameters $(1/4,1/2,1/4)$ for $\succsim^2$.

First consider the situation in which if bargaining fails both partners receive nothing. The market bargaining solution for that case suggests that partners be allocated their equilibrium bundles in the exchange economy in which each is endowed with half the resources, namely with $(1/2,1/2,1)$. Given the partners' Cobb-Douglas preferences, the market bargaining solution allocates $(2/3,2/5,4/5)$ to the first partner and $(1/3,3/5,6/5)$ to the second. This will in fact be the solution to every problem that assigns the partners with identical bundles in case bargaining fails.

To understand how a solution depends on parties' disagreement results we compare the above situation to that in which if bargaining fails the first partner ends up with the bundle $(0.25,0,0.5)$ and the second with $(0.5,0.5,0)$. The market bargaining solution for that problem entails that the partners be allocated their equilibrium allocations in the exchange economy in which the first partner is endowed with $(0.375,0.25,1.25)$ and the second partner with $(0.625,0.75,0.75)$. Solving the equilibrium for these parameters implies that according to the market bargaining solution the first partner will be allocated the bundle $(6/11,2/7,4/7)$ and the second will be allocated $(5/11,5/7,10/7)$. As can be seen, the first partner's allocation under the given circumstances is worse than his or her allocation in the first, symmetric case. The reason is that the first partner entered the bargaining situation with a disagreement bundle that is worse than that of the second partner, thus in an inferior position to his or her own position in the symmetric bargaining situation.

The difference between the partners' allocations in the symmetric versus the asymmetric situations stands in sharp contrast to the results obtained under classic bargaining solutions. To further illustrate the difference, suppose that in the above example the two partners hold identical Cobb-Douglas preferences, characterized by parameters $(1/2, 1/3, 1/6)$. When disagreement bundles are identical (as in the all-zeros disagreement point, for instance,) then as in all classic bargaining solutions the market bargaining solution will yield them equal bundles. Now, however, assume that the bargaining situation is such that in case bargaining fails the first partner gets all the land and all the wheat seeds while the second one gets nothing.[3] Any utility-based solution cannot distinguish between the two problems since both partners' default utilities, in both bargaining stories (i.e. the symmetric and the seemingly asymmetric one), are zero. Thus, any utility-based solution will still prescribe an even distribution of resources. Yet, we argue that modelling that situation as one in which both partners have identical disagreement results ignores an important aspect of the problem. After all, the partner with the land and wheat invested more in the partnership to begin with. This partner may therefore be unwilling to enter a partnership in which this larger investment is not recognized. In contrast to the utility-based approach, the market bargaining solution will distinguish between the two starting positions. Under that solution, the resolution assigned would be the equilibrium allocation in the exchange economy in which the first partner is endowed with $(1, 0.5, 2)$ and the second partner with $(0, 0.5, 0)$. The market resolution to this dispute will therefore consist of allocating one sixth of the resources to the second partner, and five times as much to the first partner.

3 Proof of Theorem 1

3.1 Proving that (i) implies (ii)

Let φ be a bargaining solution to bargaining problems that conform to the structural assumption **A0**, and suppose that φ satisfies assumptions **A1-A3**.

Step 1: It is first shown that if a binary relation $\succsim^*$ conforms to A0(2) and is unprejudiced, then $\succsim^*$ must be a linear relation, satisfying monotonicity. That is to say, there is a vector of coefficients $m \in \mathbb{R}_+^L$ such that $\succsim^*$ ranks bundles $y \in \mathbb{R}_+^L$ according to the utility function $u_*(y) = m \cdot y$.

Lemma 1. *Suppose an unprejudiced binary relation $\succsim^*$, that conforms to A0(2). Then there is a vector $m \in \mathbb{R}_+^L$ such that $y \succsim^* z$ if and only if $m \cdot y \geq m \cdot z$.*

[3] These disagreement bundles do not satisfy the condition in A0(1), but the same essential point can be made with *almost* all the land, and *almost* all the wheat seeds given to the first partner.

Proof. It is first proved that each indifference curve of $\succsim^*$ is a hyperplane. Let $t >> 0$ and consider a tangent hyperplane to the indifference curve of $\succsim^*$ that goes through t. Denote its characterizing coefficients by $m \in \mathbb{R}^L_+$ (a non-negative such tangent exists on account of A0(2b)). Note that given that $\succsim^*$ satisfies A0(2) (and thus satisfies in particular convexity) it holds that $t \succsim^* z$ for every $z \in \mathbb{R}^L_+$ for which $m \cdot z = m \cdot t$. Suppose on the contrary that the $\succsim^*$-indifference curve going through t is not a hyperplane, so that there is $y \in \mathbb{R}^L_+$ such that both $y \sim^* t$ and $m \cdot y > m \cdot t$.

Let λ be a number greater than 1. Since $m \cdot y > m \cdot t$, there is $\varepsilon(\lambda) > 0$ such that (i) $\varepsilon(\lambda)\lambda m \cdot t = (\lambda - 1)m \cdot (y - t)$, and (ii) $\varepsilon(\lambda) \to 0$ as $\lambda \to 1^+$. Set, $z = \lambda(1 + \varepsilon(\lambda))t + (1 - \lambda)y$, which for λ close enough to 1 satisfies $z \in \mathbb{R}^L_+$. Since $m \cdot z = m \cdot t$, it follows that $t \succsim^* z$.

Note that $(1 + \varepsilon(\lambda))t$ is on the interval connecting y and z. More precisely, $(1 + \varepsilon(\lambda))t = (1 - \frac{1}{\lambda})y + \frac{1}{\lambda}z$. Let $\lambda = \frac{n}{k}$ be a rational number, with $n > k > 0$ being integers. Letting $\varepsilon = \varepsilon(\frac{n}{k})$ we obtain,

$$(1 + \varepsilon)t = (1 - \frac{k}{n})y + \frac{k}{n}z \ . \tag{1}$$

We now construct allocation $b \in \mathcal{A}_n(nt(1 + \varepsilon))$ by assigning $b(i) = z$ for $i = 1, ..., k$ and $b(i) = y$ for $i = k + 1, ..., n$. Allocation b is indeed in $\mathcal{A}_n(nt)$, because by Eq. (1), $kz + (n - k)y = nt(1 + \varepsilon)$. We compare b to the constant allocation $a \in \mathcal{A}_n(nt(1 + \varepsilon))$, assigning $a(i) = t(1 + \varepsilon)$, for every $i = 1, ..., n$.

Recall that $t \sim^* y$ and $t \succsim^* z$. According to A0(2b), $\succsim^*$ is monotonic, and due to $t >> 0$, we obtain $a(i) = t(1 + \varepsilon) \succ^* t \succsim^* b(i)$, for every $i = 1, ..., n$. This is a contradiction to the assumption that $\succsim^*$ is unprejudiced. We conclude (after applying also the continuity of the relation) that $\succsim^*$ generates hyperplane indifference curves.

It remains to show that the hyperplane indifference curves are parallel. Suppose on the contrary two indifference curves which are not parallel. Let $t_1 >> 0$ lie on the lower curve, and let t_2 be the point on the second indifference curve which lies on the ray from the origin that crosses t_1, namely $t_2 = (1 + \alpha)t_1$ for some $\alpha > 0$.

As the two hyperplane indifference curves in question are non-parallel, there is a vector v such that both $t_2 + v$ and $t_2 - v$ are in the hyperplane of t_2, while $t_1 + v$ is above and $t_1 - v$ is below the hyperplane of t_1. In terms of preferences it implies that $z_2 := t_2 + v \sim^* t_2$, while $z_1 := t_1 - v \prec^* t_1$. Denote $t := (t_1 + t_2)/2$. Note that $2t = z_1 + z_2$, implying that (z_1, z_2) is a split of $2t$ into two. Furthermore, $z_2 \sim^* t_2$ but $t_1 \succ^* z_1$. Due to continuity and monotonicity there is $\lambda > 0$ such that $w := z_1 + \lambda t \sim^* t_1$. We thus obtain, $w + z_2 = z_1 + \lambda t + z_2 = (2 + \lambda)t$.

We now construct an allocation $b \in \mathcal{A}_2(2t)$: $b(1) = w - (\lambda/2)t$ and $b(2) = z_2 - (\lambda/2)t$. Note that $b(1)$ is strictly smaller than w and $b(2)$ is strictly smaller than z_2, and therefore $w \succ^* b(1)$ and $z_2 \succ^* b(2)$. By comparing allocation b to the allocation $a \in \mathcal{A}_2(2t)$, composed of $a(1) = t_1$

and $a(2) = t_2$, a contradiction is inflicted upon the assumption that $\succsim^*$ is unprejudiced. This leads us to assert that $\succsim^*$ is characterized by parallel hyperplane indifference curves, namely there is $m \in \mathbb{R}^L_+$ (where nonnegativity of the components of m stems from monotonicity of the relation) such that for every two bundles y and z, $y \succsim^* z$ if and only if $m \cdot y \geq m \cdot z$. ■

Denote a disagreement point by $d = (d(1), \ldots, d(n))$ with $d(i) = (d_1(i), \ldots, d_L(i))$ being the bundle allocated to agent i in case of disagreement. Consider a bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$, satisfying **A0**, and let a denote an allocation in its solution. By Impartial Arbitration (**A2**) there exists a relation $\succsim^*$ that arbitrates a in $(x, (\succsim^i)_{i=1}^n, d)$. Namely, $\succsim^*$ is unprejudiced, perceives a as fair given d, conforms to A0(2a)-A0(2d), and for every other allocation $a' \in \mathcal{A}_n(x)$ which $\succsim^*$ perceives as fair given d, $a(i) \succsim^i a'(i)$ for every $i = 1, \ldots, n$. According to Lemma 1, $\succsim^*$ is a linear preference. Namely, there is a vector of coefficients $m = (m_1, \ldots, m_L)$ such that for any two bundles y and z, $y \succsim^* z$ if and only if $m \cdot y \geq m \cdot z$. As $\succsim^*$ is monotone, $m \geq 0$.

The linear preference $\succsim^*$ perceives a as fair given d. Namely, there exist bundles $t_i \in \mathbb{R}^L_+, i = 1, \ldots, n$, such that $a(i) \sim^* d(i) + t_i$ for all i, and for any i, j, $t_i \sim^* t_j$. Simple arithmetics yields that for every i, $m \cdot t_i = m \cdot \dfrac{(x - \sum_{j=1}^n d(j))}{n}$, therefore, $m \cdot a(i) = m \cdot d(i) + m \cdot \dfrac{(x - \sum_{j=1}^n d(j))}{n}$. Denote $w = m \cdot \dfrac{(x - \sum_{j=1}^n d(j))}{n}$.

Step 2: Fix an agent i, and let $y \in \mathbb{R}^L_+$ be a bundle that satisfies $x \geq y$ and $m \cdot y - m \cdot d(i) = w$. Note that in this case it holds that $m \cdot (x - y) = (n-1)w + \sum_{t \neq i} m \cdot d(t)$. Define:

$$\begin{aligned} \lambda &= \frac{(n-1)w}{(n-1)w + \sum_{t \neq i} m \cdot d(t)} \\ \theta_j &= \frac{m \cdot d(j)}{\sum_{t \neq i} m \cdot d(t)}\ , \quad j \neq i \end{aligned}$$

and consider the allocation $a' \in \mathcal{A}_n(x)$ that assigns $a'(i) = y$ and $a'(j) = \dfrac{\lambda(x - y)}{n-1} + \theta_j(1 - \lambda)(x - y)$ to each $j \neq i$. Multiplying each of these bundles, for each $j \neq i$, by m, delivers $w + m \cdot d(j)$. The relation $\succsim^*$ therefore perceives a' as fair given d, and by Impartial Arbitration, $a(i) \succsim^i a'(i) = y$. It follows that $a(i) \succsim^i y$ for any bundle y that satisfies $x \geq y$ and $m \cdot y = m \cdot d(i) + w$.

Step 3: Consider the doubled problem, $(2x, ((\succsim^i)_{i=1}^n, (\succsim^i)_{i=1}^n), (d, d))$, consisting of allocating twice the resources to a double-sized group of agents in which there are two clones of every preference $\succsim^i$, $i = 1, \ldots, n$, each with a disagreement bundle identical to that in the original problem. According to Replication Invariance (**A3**) the allocation giving $a(i)$ to every i-clone is included in the solution to the doubled problem, and is arbitrated by the same relation $\succsim^*$. Denote the doubled allocation by a^d. The conclusion of the previous step is now applied to the doubled problem. It results that for every agent i and any bundle y such that $2x \geq y$ and $m \cdot y = m \cdot d(i) + w$, $a(i) \succsim^i y$.

By iterating the same argument (with $4x, 8x$, and so on) it may be concluded that any agent i prefers $a(i)$ over any bundle y such that $m \cdot y = m \cdot d(i) + w$. In other words, we obtain that for every bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$ and every agent i, any allocation in $\varphi(x, (\succsim^i)_{i=1}^n, d)$ maximizes i's utility among all the bundles whose worth w.r.t. m is the same as the worth of $d(i) + \frac{(x - \sum_{j=1}^n d(j))}{n}$, for some such vector of nonnegative coefficients $m = (m_1, \ldots, m_L)$. This is precisely the formulation of the constrained optimization problem of a pure exchange economy where each agent i is initially endowed with $d(i) + \frac{1}{n}\left(x - \sum_{i=1}^n d(i)\right)$, and the competitive equilibrium price vector is m.

Step 4: We show here that a vector m as above does exist. By **A0** (see Arrow and Debreu [1]) there exists a competitive equilibrium for the above exchange economy. Thus, a vector m exists as desired – it is a prevailing competitive price system. The allocation a is an equilibrium allocation corresponding to the equilibrium prices given by $(m_1, \ldots, m_L)$, under endowment of $d(i) + \frac{(x - \sum_{j=1}^n d(j))}{n}$ for agent i, for all i. Following the Equivalence Principle (**A1**), all the equilibrium allocations that correspond to the prices $(m_1, \ldots, m_L)$ are included in the solution. The next claim shows that these are the only allocations included in the solution.

Claim 1. *Suppose that $a \in \varphi(x, (\succsim^i)_{i=1}^n, d)$ is an equilibrium allocation corresponding to an equilibrium price vector m, for endowments $d(i) + \frac{1}{n}\left(x - \sum_{i=1}^n d(i)\right)$ per agent i. Then for any $a' \in \varphi(x, (\succsim^i)_{i=1}^n, d)$, a' is also an equilibrium allocation in the same exchange economy, that corresponds to the same price vector m.*

Proof. Let $a = (a(i))_{i=1}^n$ and $a' = (a'(i))_{i=1}^n$ be two allocations in $\varphi(x, (\succsim^i)_{i=1}^n, d)$, and suppose that a is an equilibrium allocation in an exchange economy with the above endowments, corresponding to the price vector m. The Equivalence Principle (**A1**) entails that all the agents are indifferent between the two allocations: $a(i) \sim^i a'(i)$ for every i.

Since a is an equilibrium allocation, $a(i)$ - the share of i - optimizes i's utility subject to i's budget constraint. Suppose that there is an agent j whose allocation $a'(j)$ satisfies $m \cdot a'(j) - m \cdot d(j) > m \cdot a(j) - m \cdot d(j)$. As $\succsim^*$ perceives a as fair given d, there must be another agent i whose allocation $a'(i)$ satisfies $m \cdot a'(i) - m \cdot d(i) < m \cdot a(i) - m \cdot d(i)$. This would make $a'(i)$ an allocation that does not exhaust i's budget constraint in the above economy, yet is equivalent to $a(i)$ (by the Equivalence Principle, as explained above). By **A0**, all the agents' preferences are monotonic. Thus agent i could improve on $a(i)$, within the limit of his or her budget constraint, by adding a strictly positive bundle to $a'(i)$. It would follow that $a(i)$ is not optimal for i under i's budget constraint, contradicting the assumption that $a(i)$ is an equilibrium allocation as specified above. Therefore, we must conclude that $m \cdot a'(i) - m \cdot d(i) = m \cdot a(i) - m \cdot d(i)$ for every i. It follows that $a'(i)$ is optimal for every i under the budget constraint corresponding to m, making the entire allocation a' an equilibrium allocation in the exchange economy with endowments $d(i) + \frac{1}{n}\left(x - \sum_{i=1}^{n} d(i)\right)$, $i = 1, \ldots, n$, under the price vector m. ■

We conclude that, given a bargaining problem, a solution φ to this problem equals the set of all the equilibrium allocations in the exchange economy where each agent i is endowed with $d(i) + \frac{1}{n}\left(x - \sum_{i=1}^{n} d(i)\right)$, corresponding to some vector of equilibrium prices m.

Step 5: The last step in the proof consists of asserting that there exists a selection of equilibria across different problems that is well defined, and satisfies replication consistency. For that matter, a bargaining problem is a *replicated problem* if it is a k-replication (for some $k \in \mathbb{N}$) of another bargaining problem.

For any bargaining problem which is not replicated, the selection of an equilibrium price vector is unconstrained. Any equilibrium price vector may be selected, and all of its corresponding equilibrium allocations are assigned as a resolution. Given that selection, Replication Invariance (**A3**) dictates the selection in replicated problems. These choices are well-defined as any replicated problem cannot be obtained from two different problems with different equilibria selections. That is, it can always be traced to the minimal number of replications, and by Replication Invariance (**A3**), the equilibrium price vector choice is consistent for any number of replications. Obviously, the suggested choice maintains replication consistency.

3.2 Proving that (ii) implies (i)

Suppose that (ii) of the theorem holds.

In order to prove **A1**, observe first that all the allocations corresponding to the same vector of equilibrium prices are indifferent for all agents. For the other part of **A1**, suppose that $a = (a(i))_{i=1}^{n}$ and $a' = (a'(i))_{i=1}^{n}$ are two allocations between which all agents are indifferent: $a(i) \sim^i a'(i)$ for every i. Assume that a is an equilibrium allocation corresponding to an exchange economy with endowments $d(i) + \dfrac{1}{n}\left(x - \displaystyle\sum_{i=1}^{n} d(i)\right)$, $i = 1, \ldots, n$. Consequently, according to the proof of Claim 1, a' is an equilibrium allocation for the same prices, and thus either both are assigned by a solution or none is.

Impartial Arbitration (**A2**) is implied since for any equilibrium allocation a under endowments $d(i) + \frac{1}{n}(x - \sum_{i=1}^{n} d(i))$ the corresponding vector of equilibrium prices p can be extended to a linear preference relation, which is unprejudiced as a simple implication of its definition, and assigns the same value $p \cdot a(i) - p \cdot d(i) = p \cdot \frac{1}{n}(x - \sum_{i=1}^{n} d(i))$ to every i. If an allocation $a' \in \mathcal{A}_n(x)$ is perceived as fair given d by this linear preference, then it means that for a' as well, $p \cdot a'(i) = p \cdot d(i) + p \cdot \frac{1}{n}(x - \sum_{i=1}^{n} d(i))$, and therefore it must hold that i (weakly) prefers her or his equilibrium allocation $a(i)$ to $a'(i)$.

For Replication Invariance (**A3**), the fact that replications of allocations in a solution are included in the solution to the replicated problem, follows from the assumption that a solution is replication-consistent. This restricts a solution to choose the same market equilibrium for a problem and its replications. On that account, each allocation in the solution to the replicated problem is perceived as fair by the same linear preference defined by the chosen vector of equilibrium prices, and any other allocation which satisfies the budget constraint generated by these prices is Pareto-dominated by the corresponding equilibrium allocations which compose the selected solution.

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