Competitive equilibrium as a bargaining solution: an axiomatic approach*

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Abstract

The paper introduces an axiomatic characterization of a solution to bargaining problems. The bargaining problems addressed are specified by: (a) the preference relations of the bargaining parties (b) resources that are the subject of bargaining, and (c) a pre-specified disagreement bundle for each party that would result in case bargaining fails. The approach is ordinal in that parties' preferences are over bundles of goods and do not imply any risk attitudes. The resulting solution is accordingly independent of the specific utility indices chosen to represent parties' preferences. We propose axioms that characterize a solution that matches each bargaining problem with an exchange economy whose parameters are derived from the problem, and assigns the set of equilibrium allocations corresponding to one equilibrium price vector of that economy. The axioms describe a solution that is the result of an impartial arbitration process, expressing the view that arbitration is a natural method to settle disputes in which agents have conflicting interests, but can all gain from a compromise.

Keywords: Bargaining, exchange economy, ordinal bargaining solution, competitive equilibrium.
JEL classification: C70; C78; D51; D58

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1 Introduction

In many business disputes the source of controversy between the conflicting parties concerns the distribution of assets among them. One of the most common methods for dispute resolution, alternative to litigation, is arbitration. When the conflicting parties choose to undergo arbitration they often reach quicker, more efficient, and sometimes less costly resolutions, compared to those they would have reached through conventional court proceedings. In contrast to litigation, which can be initiated whenever one of the conflicting parties decides to resort to court, a key feature of arbitration is that it can be initiated only upon the unanimous consent of the parties involved, and conducted according to principles they all agree on.

When a dispute over assets arises each party has a default allocation that can be achieved even when no agreement is reached. Surely a default of no assets can always be achieved, but in some cases, when a clear ownership of assets is involved, for instance when one of the disputing parties owns a real estate property, a more sophisticated disagreement result obtains. Nonetheless, in case of disagreement parties usually end up in a less favorable position, whereby the prolongation of the conflict delays the distribution of assets (or at least of those assets that remain after the parties attain their defaults). This could be the case, for instance, if the dispute ends up in a lengthy court litigation.

The current paper addresses disputes over multiple assets, involving several parties. Disputes are described by a bundle of assets that should be allocated to a number of parties, and by the preferences of those parties over assets (without supposing that there is any risk involved). Preferences are only assumed to be ordinal, namely there is no notion of strength of preference and the only significant question when comparing two bundles is which of them is preferred over the other. It is supposed that each concerned party can prevent unilaterally an agreement (e.g. turn to court), in which case each of the involved parties will obtain a pre-defined default bundle. The problems handled are non-trivial in the sense that all parties maintain that the dispute may be resolved in more beneficial ways compared to their default results, hence all parties involved have an incentive to compromise. On the other hand, each party retains the option to influence the obtained compromise by imposing unilaterally the inferior default result. A solution is sought, that would assign to each dispute allocations of the assets under consideration.

Given that each entity involved in a dispute can cause the breakdown of an agreement, a solution would provide a viable method of dispute resolution only if parties are persuaded to accept it. Finding such a solution is our goal in the current essay, which offers an axiomatic characterization of dispute resolution.

When entering a business relationship, parties often sign a contractual agreement that requires them to resolve possible future disputes by means of an arbitration process. The contract furthermore specifies the principles by which arbitration will take place. Analogously to such a contract, our axioms formulate arbitration principles, describing a solution that results from an impartial arbitration process. We then show that a solution satisfying these axioms must be of a specific functional form (and vice versa). Namely, agreeing to an arbitration process that accords with the proposed principles is equivalent to employing the functional mechanism.

The problems we address consist of allocating goods to agents. The primary economic mechanism that is designed for that task is an exchange economy. Given initial bundles of goods, one bundle per each agent, a competitive equilibrium of the economy determines which allocation prevails: what goods each agent sells and how much each agent decides to consume. The solution concept characterized in this paper, termed *the market bargaining solution*, utilizes the market mechanism as an instrument. This is done by associating each dispute with an exchange economy, selecting an equilibrium price vector for that economy and assigning as a resolution all the equilibrium allocations that correspond to the selected equilibrium price vector (the solution therefore depends on the selection and is not unique). The exchange economy that is matched to a dispute is the one involving agents with the preferences of the disputing parties, each endowed with the corresponding party's default allocation plus an equal share of the remainder. Each concerned party is therefore considered to be in possession of his or her default result, with the remaining assets divided equally. Our axiomatic characterization establishes that equilibrium allocations can be viewed as resolutions to a dispute rendered by an impartial arbitrator. An interesting feature of the development is that equilibrium prices are derived endogenously per dispute, and interpreted as representing an impartial arbitrator.

The problems handled in this paper are similar to classic bargaining problems as formulated by Nash [12]. Both types of problems concern the distribution of resources among a group of agents. In both cases a solution determines, for each problem, how resources are to be allocated to the parties involved, where each party is assumed to be able to reject compromise and unilaterally impose an inferior result on all parties involved. The axiomatic approach, aimed at attaining a consensus among disputing parties, is prevalent in the bargaining literature. The difference between the two types of problems lies in the assumptions made on the parties' preferences over resources, and in the formulation of feasible allocations and disagreement results.

Under the classic bargaining approach problems are formulated in terms of utilities. The default result, as well as feasible allocations of resources, are described in utility terms, suppressing any reference to the actual assets that underlie those utilities. Consequently, any two distinct economic environments that generate the same utility image will inevitably be assigned the same allocation as a solution. The classic bargaining approach relies on the assumption that meaningful information regarding the cardinality of parties' preferences is available, for instance due to preferences being defined over lotteries. However, for some problems such information is not attainable. Roemer, [16] and [17], discusses the implications of assuming that bargaining problems can be formulated in terms of utilities, and suggests a more general bargaining setup that is defined in terms of resources and preferences. This is the setup we employ in this paper, and accordingly, the solution we offer is specified in terms of assets. The difference between the resource-based bargaining approach and that of the classic bargaining literature can be illustrated through two simple examples.

Imagine two business partners engaged in a dispute over one acre of land, where in case of disagreement both get nothing. Each of the partners naturally wants to have as much of the land as possible. If no cardinal information regarding the partners' preferences is available, then their preferences for more land can be represented by any strictly increasing function of x, x being the received fraction of the land. Consider two such representations, $u_1(x) = u_2(x) = x$ on one hand, and $u_1(x) = \sqrt{x}$ and $u_2(x) = 1 - \sqrt{1-x}$ on the other, where u_i stands for partner i's utility from the land. Both representations induce the same utility-possibility set, namely the same set of utility allocations (these are the points in the triangle bounded within the origin and the points $(0, 1)$ and $(1, 0)$), and the same $(0, 0)$ default result in case of disagreement. Hence, every bargaining solution concept which employs the classic bargaining setup will match both problems with the same solution, that is with the same allocation of utilities. Symmetry of the utility-possibility set would lead to a fifty-fifty distribution of utilities under all well-known bargaining solution concepts. However, while under the first utilities representation this solution translates to assigning each of the partners half the acre of land, in the second utilities representation it entails allocating $1/4$ of the land to one partner and $3/4$ to the other. By contrast, a market bargaining solution will entail an equal division of the land regardless of the choice of specific utility representations, since underlying preferences over assets and the assets available for division remain unchanged.

Aside from assuming only ordinality of preferences, another difference between the two approaches is manifested in the conversion of disagreement results into bargaining powers. To demonstrate this difference, suppose two partners with identical Cobb-Douglas preferences disputing over one acre of land and one ton of wheat seed. Imagine that in case bargaining fails the first partner gets all the land while her or his rival gets nothing (let's say, because the first partner owned the land before entering the partnership). Any utility-based solution will ascribe equal bargaining powers to both partners since their default utilities are both zero. Yet it is unlikely that parties will concede to bargaining powers being interpreted like that, as this would mean ignoring differences related to investment size and initial ownership of assets. Our approach accounts for such differences by expressing the problem, including parties' default results, in assets terms.

On the negative side, an inevitable limitation of the suggested approach is that it applies only to bargaining problems whose corresponding exchange economies possess competitive equilibria. To that end, the paper characterizes a solution under assumptions that guarantee the existence of such an equilibrium. As is the case in market-related questions in general, here as well multiple equilibrium price vectors may exist. A market bargaining solution selects one of them, subject to some consistency requirements across different problems.

1.1 Literature review

The current paper touches on three central topics in economics: Bargaining Theory, in considering bargaining problems; Competitive equilibrium, in the solution outlined; And social choice theory, in the special case of parties with equal bargaining powers. As there is an abundance of papers studying these topics and the relationships between them, we confine ourselves to discussing only the most relevant ones.

In the special case where parties enter a dispute with equal bargaining powers, the solution offered here is competitive equilibrium from equal division of resources. This is a prominent solution concept in the theory of fair allocations, and is known to satisfy many socially desirable attributes such as no envy, individual rationality, efficiency, and various consistency conditions. Competitive equilibrium from equal division was characterized axiomatically in various papers, for example Thomson, [24] and [25], Nagahisa, [10] and [11], and Nagahisa and Suh [9]. These axiomatizations can be 'plugged in' as a solution to disputes in which the disagreement point is symmetric. However, the general case, in which parties posses possibly disparate bargaining powers, is not accommodated by those papers, as naturally parties within the social choice literature are treated equally. Hence, these papers do not characterize a resolution of disputes over assets, as we present here.

As for the characterizations themselves, the fair allocations literature employs axioms that are designed to appear normatively appealing in the eyes of social planners, who commonly seek for solutions that focus primarily on traits such as fairness and stability. On the other hand, the problems addressed in this paper require the cooperation of opposing agents rather than the consent of a social planner. To achieve agents' cooperation, the axioms describe individually appealing attributes of a solution, designed to gain the consent of the agents involved. For example, the fair allocations literature employs stability requirements which guarantee that sub-groups cannot do better on their own. But requirements of that sort are not appealing for agents that are concerned only with their own allocated assets. An agent will be indifferent to any re-distribution of assets among a sub-group that does not contain him or her, while preferring a profitable re-distribution of assets among a sub-group to which the agent belongs.

This paper handles bargaining problems that are formulated in terms of assets and ordinal preferences, rather than in terms of utility values. Roemer, in [16] and [17], argues for the need of this kind of formulation, which allows for a richer variety of solution concepts. Roemer characterizes classic bargaining solution in this richer setup and demonstrates the strength of assumptions required for that characterization. Sertel and Yildiz [22] show that within the classic bargaining setup it is impossible to define a solution which for every utility image of an exchange economy assigns utilities of respective Walrasian allocations.

There are several other solutions that are characterized within an ordinal setup. Chen and Maskin [3] extend the setup to include production and characterize an egalitarian solution, equating agents' utilities, under a requirement that agents' utilities do not decrease following an improvement in technology. Perez-Castrillo and Wettstein [15] employ a condition involving contributions of agents to sub-groups, generalizing attributes of the Shapley value. They characterize a solution which generalizes the Pareto efficient egalitarian-equivalence (PEEE) solution of Pazner and Schmeidler [14] to the case of non-equal initial positions. PEEE allocations, designed as a fair solution to resource allocation problems, form a resolution to disputes that are based on ordinal preferences, as in the setup employed here, as long as the disagreement bundles are the same for all agents. These are allocations which for each agent are equivalent preference-wise to the same fixed bundle. Thus, they are preference-wise equivalent to egalitarian allocations in some hypothetical economy. A disadvantage of the PEEE solution from the perspective of disputing agents is that it may allocate to one agent a bundle which dominates, good-by-good, another agent's bundle (when there are more than two parties to a dispute). In the course of a dispute, such a finding may lead to a denial of compromise.

Another characterization of a bargaining solution in an ordinal setup is Nicolo and Perea [13]. Their paper addresses two-persons bargaining problems, also formulated by means of resources and preferences (under some conditions their development can be applied to problems with any number of agents). Their solution depends on an exogenously given increasing family of allocation sets, and their characterization hinges on a form of monotonicity with respect to enlargement of possibility sets, and on a condition that binds together problems with different preferences.

In the problems that we have in mind agents agree in advance, prior to engaging in a joint business and before any dispute arises, to employ an arbitration mechanism as depicted in the axioms. It follows that the parties to be involved in a dispute are fixed. Thus, assumptions that involve possible changes concerning the nature of the involved agents (e.g., assumptions referring to sub-groups or to transformed preferences) will be deemed less relevant by agents.

Among the papers that take a preference-based approach to bargaining is Rubinstein, Safra and Thomson [19], that describes two-persons bargaining problems in terms of preferences over lotteries. In that paper a solution is defined by means of preferences, and is shown to coincide with the famous Nash bargaining solution [12] whenever preferences admit an expected utility representation. The result of Rubinstein, Safra and Thomson is extended in Grant and Kajii [5] to additional preference types. Essentially, the preferences and solutions considered in these works are cardinal. That is to say, preferences are represented by a utility function that is invariant only under positive linear transformations (where this type of uniqueness is the result of using a setup that contains lotteries), hence the solution to a bargaining problem may change following different order-preserving representations of preferences over assets. This differs from our framework, which addresses preferences over assets which are ordinal in nature.

Our ordinal approach should not be confused with ordinal solutions to utility-formulated bargaining problems. The latter approach, as in Shapley [23] and Safra and Samet [20], among others, still considers classic bargaining problems that are described by means of agents' utility levels. Therefore, it cannot distinguish between different disputes that induce the same utilities image (as in the examples above). Another related paper, employing an economy as a solution device, is Trockel [26]. Trockel considers bargaining problems described in terms of utilities, and matches each one with an Arrow-Debreu economy the (unique) competitive equilibrium of which identifies with a corresponding asymmetric Nash bargaining solution.

Finally we mention a few well-known works in Bargaining Theory. Nash [12] was the first to formulate what is now known as the Nash Bargaining Problem: a two-persons setup in which parties can collaborate in a way that will be beneficial for both, and need to agree on the utility values that each will gain from this collaboration, otherwise they will obtain inferior disagreement values. Nash [12] phrases the problem in terms of utilities and offers an axiomatic treatment under the assumption that utility is cardinal (unique up to positive linear transformations). The axioms are shown to lead to a unique solution in terms of utilities. Later works suggest alternative axiomatizations and solutions to the Nash Bargaining Problem, still considering a utility-based formulation. Among those are Kalai and Somorodinsky [7], Kalai [6], and many others.

2 Characterization and result

2.1 Setup and notation

Suppose a set of divisible goods, $\{1, \ldots, L\}$, and consider bundles of these goods which are elements in $\mathbb{R}_+^L$. For two bundles $x, y \in \mathbb{R}_+^L$ we write $x \geq y$ whenever $x_\ell \geq y_\ell$ for every $\ell = 1, \ldots, L$, and $x >> y$ whenever $x_\ell > y_\ell$ for every $\ell = 1, \ldots, L$. The set of allocations of bundles to a group of n agents is $\mathcal{A}_n = (\mathbb{R}_+^L)^n$. An element of $\mathcal{A}_n$, termed an *allocation*, is denoted by $a = (a(1), \ldots, a(n))$, where $a(i) = (a_1(i), \ldots, a_L(i))$ is the bundle allocated to the i-th agent, containing quantity $a_\ell(i)$ of good ℓ. For given resources $x = (x_1, \ldots, x_L)$, $\mathcal{A}_n(x)$ denotes the subset of $\mathcal{A}_n$ consisting of distributions of x to n parties, namely of allocations a that satisfy, $\sum_{i=1}^{n} a_\ell(i) = x_\ell$ for every good ℓ.

The bargaining problems addressed are triplets, $(x, (\succsim^i)_{i=1}^n, d)$, consisting of resources $x \in \mathbb{R}_+^L$ that need to be split between agents $i = 1, \ldots, n$, endowed with preferences over bundles given by binary relations over $\mathbb{R}_+^L$, $(\succsim^i)_{i=1}^n$, where the disagreement point is $d = (d(1), \ldots, d(n)) \in \mathbb{R}_+^L$, $d(i)$ being the bundle to be allocated to party i in case bargaining fails. The asymmetric and symmetric parts of each $\succsim^i$ are respectively denoted by $\succ^i$ and $\sim^i$. As explained in the Introduction, in order for the market bargaining solution to be well defined it is essential that an exchange equilibrium exists in any problem considered. To guarantee this, a structural assumption is employed.

A0. Structural assumption.
Any bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$ satisfies the following assumptions:

(1) $x >> \sum_{i=1}^{n} d(i)$

(2) For every i and $\succsim^i$ over $\mathbb{R}_+^L$ it holds that:

(a) $\succsim^i$ is complete and transitive.

(b) $\succsim^i$ is monotonic: for any $y, z \in \mathbb{R}_+^L$, if $y >> z$ then $y \succ^i z$.

(c) $\succsim^i$ is continuous: for every $y \in \mathbb{R}_+^L$ the sets $\{z \in \mathbb{R}_+^L \mid z \succsim^i y\}$ and $\{z \in \mathbb{R}_+^L \mid y \succsim^i z\}$ are closed.

(d) $\succsim^i$ is convex: for every $y, z \in \mathbb{R}_+^L$, if $y \succ^i z$ then $\lambda y + (1-\lambda)z \succ^i z$ for every $\lambda \in (0, 1)$.

(e) $\succsim^i$ induces differentiable indifference curves: for $y \in \mathbb{R}_+^L$, $y >> 0$, define $D(y) = \{d \in \mathbb{R}^L \mid \exists \varepsilon > 0 \text{ s.t. } y + \varepsilon d \in \mathbb{R}_+^L \text{ and } y + \varepsilon d \succ^i y\}$. Then there is $v_y \in \mathbb{R}^L$ such that $D(y) = \{d \in \mathbb{R}^L \mid d \cdot v_y > 0\}$ (this definition is due to Rubinstein, [18]).

The conditions imposed on agents' preferences described in (2)(a) through (2)(d) are the standard Arrow-Debreu [1] conditions. The less standard part of the assumption, (2)(e), is in fact not required for the existence of an equilibrium but rather for the necessity part of our theorem (where the axioms are shown to be implied by the solution). The assumption in (1), combined with monotonicity (**A0**(2b)), implies that problems are non-trivial, in the sense that some allocations are deemed by everybody as better than the default. Technically speaking, this additional assumption guarantees that an equilibrium exists for the economy that corresponds to any extended problem.

A *bargaining solution* is a correspondence φ that assigns to every bargaining problem that satisfies the structural assumption above, a nonempty set of distributions of x to the agents $i = 1, \ldots, n$. That is, $\varphi(x, (\succsim^i)_{i=1}^n, d)$ is a subset of $\mathcal{A}_n(x)$.

2.2 Assumptions and main result

Four attributes are assumed on a bargaining solution φ. The first one is that any allocation assigned by φ is Pareto optimal. Any solution that does not comply with the Pareto condition can obviously be concertedly improved by the parties involved, therefore Pareto optimality constitutes a fundamental aspect in the plausibility of the solution.

A1. Pareto.
Let $(x, (\succsim^i)_{i=1}^n)$ be a bargaining problem. For every allocation $a \in \varphi(x, (\succsim^i)_{i=1}^n)$ and any allocation $a' \in \mathcal{A}_n(x)$, if $a'(i) \succsim^i a(i)$ for all parties $i = 1, \ldots, n$, then $a'(i) \sim^i a(i)$ for all $i = 1, \ldots, n$.

Agents in our framework are fully described by their preferences, and each agent is supposed to be concerned only with his or her own wellbeing. Thus, if two allocations yield for an agent bundles between which the agent is indifferent, then those two allocations will be regarded by the agent as equivalent. The second assumption maintains that all the agents deem all the allocations assigned by a given solution as equivalent in that sense. The solution therefore satisfies the property of *single-valuedness*, in the terminology of Moulin and Thomson [8]. Namely, the solution is, preference-wise, a singleton. If this were not the case, agents could not be expected to agree on any particular allocation assigned by a solution, since each allocation assigned could yield them a different utility level. The axiom further states that if two allocations are equivalent in any relevant respect (i.e., considered equivalent by any of the agents), either both of them are assigned by the solution or none of them is. A solution is therefore *a full correspondence*, as defined in Roemer [17].

A2. Equivalence Principle.
Suppose a bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$ and two allocations, $a, a' \in \mathcal{A}_n(x)$. If $a, a' \in \varphi(x, (\succsim^i)_{i=1}^n, d)$ then $a(i) \sim^i a'(i)$ for every i. In the other direction, if, for every i, $a(i) \sim^i a'(i)$, then $a \in \varphi(x, (\succsim^i)_{i=1}^n, d)$ if and only if $a' \in \varphi(x, (\succsim^i)_{i=1}^n, d)$.

We turn now to discuss our main axiom, which depicts the solution as the result of an impartial arbitration process.

A bargaining situation demands discretional agreement of all the agents, otherwise the disagreement point, unfavorable to all agents, will prevail. The need for unanimous consent among agents creates tension between their incentive to compromise and the power held by each agent to threaten the others with the breakdown of negotiations. An involvement of a third party is often required, and welcomed by agents, as a way to resolve this tension. A prevalent dispute resolution method, administered by a third party, is arbitration. In many business partnerships parties agree in advance that any future dispute among them will be resolved by arbitration, and specify beforehand its fundamentals.

Our third axiom requires that any allocation assigned by a solution be the result of arbitration conducted according to three basic principles. Hence, agents subscribing to this axiom accept this type of arbitration as a dispute resolution method. In conformity with the representation of parties to a dispute by their preferences, an arbitrator in our framework is modelled by a preference relation, satisfying (as do preferences of disputing parties) assumption A0(2). Any ranking of alternatives by such an arbitrating preference relation should be understood as reflecting the principles that guide the decisions of an arbitrator rather than an expression of this arbitrator's personal preferences.

An arbitrator will be interested in the allocation of resources among parties and not in single bundles per se. Accordingly, the requirements involving an arbitrating preference pertain to allocations as a whole. The requirements specify guidelines for the arbitral decision-making. Three standards are conveyed in our third axiom: (1) Arbitration should be unprejudiced, in the sense that no allocations should be eliminated a-priori; (2) Arbitration should treat all disputing parties fairly; (3) The arbitrator can be trusted to execute the solution through bilateral negotiations with the disputing parties.

The three definitions below correspond to these three standards of arbitration. The axiom to follow applies these definitions to a preference relation that serves as an arbitrator.

The first definition identifies a preference relation as unprejudiced whenever no allocation of resources to a number of parties is dominated by another allocation of the same resources to the same number of parties.

Definition 1. A binary relation $\succsim^*$ is *unprejudiced* if for every $y \in \mathbb{R}_+^L$ and $k \in \mathbb{N}$, and every $a, b \in \mathcal{A}_k(y)$, it cannot be the case that $a(i) \succ^* b(i)$ for every $i = 1, \ldots, k$.

When a preference relation reflects the decisions of an arbitrator, having one allocation dominated by another entails that the dominated allocation will never be assigned by the arbitrator. A dominated allocation will therefore be censored by the arbitrator before even considering a bargaining problem. Allocations, however, should be eliminated or chosen only upon observing the preferences of the parties involved in a dispute (for instance Pareto dominated allocations will be eliminated after observing parties' preferences), otherwise solutions may be renounced prior to approaching a dispute, even if they may turn out to be favorable to the parties involved.

The second standard of arbitration requires that an arbitrator rule fairly in a dispute. To define what fair ruling means in our setup, recall that parties may start off with different fallback options. For instance, one agent may be in possession of some of the assets while others are not. In such a case this agent will naturally expect her or his ownership of assets to be taken into account in the arbitral decision, perceiving settlements of the dispute which ignore different a-priori rights as unfair. Our definition of fairness thus depends on the disagreement point. It states that a preference relation perceives an allocation as fair given a disagreement point whenever the surplus generated for each party, on top of this party's default result, is judged as equivalent by the relation.

Definition 2. A binary relation $\succsim^*$ *perceives an allocation* $a \in \mathcal{A}_n$ *as fair given a disagreement point* d if there exists a bundle $t \in \mathbb{R}_+^L$ such that for all i, $a(i) \sim^* d(i) + t$.

Definition 2 allows us to express the most basic standard of arbitration, namely an impartial treatment of parties. In the special case where agents' claims are equal, agents will expect an impartial arbitrator to rule fairly and assign allocations which the arbitrator deems equitable. Stated differently, agents will not respect the authority of an arbitrator who intends to assign better bundles to their peers. Generalizing this intuition, when different agents have different starting positions, these will have to be respected by the arbitrator. However, once those starting positions are taken into account, agents will expect an impartial treatment. Namely, any addition on top of these starting positions should be equitable.

The third definition describes consistency of the solution under bilateral negotiations of the disputing parties with another relation, denoted $\succsim^*$. Such consistency holds when each party is allocated equivalent bundles both as a solution to the original problem, and when negotiating with the relation $\succsim^*$ alone. In the axiom to follow this definition is applied to an arbitrating preference $\succsim^*$, to express that our solution mechanism is consistent under bilateral negotiations with the arbitrator. The definition furthermore requires that $\succsim^*$ judge as fair any allocation within the bilateral negotiations. When this requirement is imposed on an arbitrating preference, its violation means that the arbitrator demonstrates bias when negotiating separately with the bargaining parties. In that case, agents cannot trust the arbitrator to settle the dispute in an impartial manner. A violation of this requirement is therefore likely to incite parties' objection to the arbitration process. Altogether, when an arbitrator is able to bilaterally negotiate a solution to a problem as specified in the definition, parties do not object to negotiating with the arbitrator alone, as they know they will obtain an equivalent bundle, and at the same time trust the arbitrator to maintain impartiality when negotiating with others. Similarly, parties know that their peers have no reason to object to such separate negotiations. Hence parties are not worried by their peers rejecting the authority of the arbitrator and imposing the inferior default solution.

Some notation is required to formulate the definition. Let $(x, (\succsim^i)_{i=1}^n, d)$ be a bargaining problem. Then the *replaced problem* $(x, (\succsim^i, \underbrace{\succsim^*, \ldots, \succsim^*}_{n-1 \text{ times}}), d)$ is derived from it by letting $\succsim^*$ replace all agents $j \neq i$, including their disagreement bundles. An allocation $a \in \varphi(x, (\succsim^i, \underbrace{\succsim^*, \ldots, \succsim^*}_{n-1 \text{ times}}), d)$ still assigns bundle $a(i)$ to agent i.

Definition 3. A binary relation $\succsim^*$ *can bilaterally negotiate* $a \in \varphi(x, (\succsim^i)_{i=1}^n, d)$ if for any agent i, and any allocation $a' \in \varphi(x, (\succsim^i, \underbrace{\succsim^*, \ldots, \succsim^*}_{n-1 \text{ times}}), d)$, $a(i) \sim^i a'(i)$, and $\succsim^*$ perceives a' as fair given d.

The next axiom, Impartiality, describes a solution as the result of an impartial arbitration process, by stating that a solution to a dispute admits an arbitrating preference that fulfills all three standards specified above. Thus, Impartiality implies that the solution is a fair resolution of the dispute, negotiated by an unprejudiced arbitrator. Furthermore, the arbitrator can implement the solution through separate negotiations with the bargaining parties.

When disputing parties adopt Impartiality they commit to resolving disputes by means of an arbitration conducted in line with the three principles listed above. The parties agree on an arbitrator chosen from among candidates who approach the problem open-mindedly.[1] They expect that the arbitrator be just and treat all parties equitably, once their different initial

[1] In real-life disputes, arbitrators are frequently chosen from among members of a specialized institution, such as the American Arbitration Association (AAA), so that parties can trust their impartiality. As stated on the AAA website, "These neutrals are bound by AAA established standards of behavior and ethics to be fair and unbiased".

rights are taken into account. Finally, parties agree and trust the arbitrator to negotiate the solution through separate discussions with each of them.

A3. Impartiality.
Let $(x, (\succsim^i)_{i=1}^n, d)$ be a bargaining problem and $a \in \varphi(x, (\succsim^i)_{i=1}^n, d)$ an allocation in its solution. Then there is an unprejudiced binary relation that conforms to A0(2), which perceives a as fair given d and can bilaterally negotiate it.

Note that we do not require that an arbitrator negotiating a solution to a problem be unique. Nevertheless, our theorem will ultimately imply such uniqueness. Although there may be multiple adequate arbitrators per problem, the characterized solution selects one arbitrator,the only one that fits the allocations assigned as a solution to that problem.

The last axiom is a consistency assumption referring to a certain form of re-scaling. It compares between solutions to two problems, an original problem and its replication. By the latter we mean the problem in which both the number of agents with their starting positions, and the resources to be allocated, are replicated by the same factor, so that the replicated problem consists of allocating k times the original resources to a group of agents comprised of k copies of each original agent, each with her or his original default bundle. By having multiple representatives of each original preference we create a situation whereby each agent's claims are replicated. The resources in dispute are replicated as well in the same manner. The combined effect of both is akin to that of re-scaling, and Replication Invariance asserts that a solution is invariant under this sort of re-scaling. First, the re-scaled solution is contained in the solution to the re-scaled problem, hence, by the Equivalence Principle (**A2**), the solutions to the original and to the re-scaled problems are preference-wise equivalent. Second, if an arbitrator can serve in the original problem then he or she can also serve in its re-scaling. This condition is similar to an attribute by the same name that was introduced by Thomson (see the survey [25] and the references therein).

A4. Replication Invariance.
Let $(x, (\succsim^i)_{i=1}^n, d)$ be a bargaining problem. Then:

(a) If $a \in \varphi(x, (\succsim^i)_{i=1}^n, d)$ then $a^k = (\underbrace{a, \ldots, a}_{k \text{ times}}) \in \varphi(kx, (\underbrace{(\succsim^i)_{i=1}^n, \ldots, (\succsim^i)_{i=1}^n}_{k \text{ times}}), (\underbrace{d, \ldots, d}_{k \text{ times}}))$.

(b) If an unprejudiced binary relation that conforms to A0(2) perceives $a \in \varphi(x, (\succsim^i)_{i=1}^n, d)$ as fair given d, and can bilaterally negotiate it, then it can also bilaterally negotiate

$$a^k \in \varphi(kx, \underbrace{((\succsim^i)_{i=1}^n, \ldots, (\succsim^i)_{i=1}^n)}_{k \text{ times}}, \underbrace{(d, \ldots, d)}_{k \text{ times}}).$$

Our main theorem shows that under the structural assumption (**A0**), axioms **A1-A4** characterize a market bargaining solution. Among these axioms, Pareto (**A1**) is a fundamental condition without which a solution is simply senseless, and the bargaining parties will never accept its assigned allocations. Replication Invariance (**A4**) and the Equivalence Principle (**A2**) guarantee the consistency of a solution from two aspects. The first aspect is consistency with respect to re-scaling, and the second is consistency with respect to the representation of a problem, whereby a solution depends entirely on preferences. These three axioms are standard in the literature. The primary assumption in the characterization of a solution is Impartiality (**A3**), which describes a solution as the result of an impartial arbitration process. Our theorem states that parties' consent to the arbitration principles encapsulated in the above axioms is equivalent to allotting each party with her or his default bundle plus an equal share of the remaining resources, and allowing parties to trade among themselves.

Note that the dispute resolution that we offer is not unique, as multiple equilibrium price vectors may exist per dispute. The axiomatization does not in general mandate that any specific equilibrium price vector of the induced exchange economy be selected. It only dictates that the selection is consistent in the two following manners.

Definition 4. A bargaining solution φ is *replication-consistent* if the equilibrium allocations it assigns, given any bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$ and its replication $(kx, \underbrace{((\succsim^i)_{i=1}^n, \ldots, (\succsim^i)_{i=1}^n)}_{k \text{ times}}, \underbrace{(d, \ldots, d)}_{k \text{ times}})$ $(k \in \mathbb{N})$, correspond to the same vector of equilibrium prices.

Definition 5. A bargaining solution φ is *replacement consistent* if whenever it assigns allocations that correspond to a vector p of equilibrium prices, given a bargaining problem, then the allocations it assigns to the replaced problem that is created when all agents but one are replaced with a p-linear preference[2], correspond to the same vector of equilibrium prices p.

Replication consistency is important in order to avoid cases where essentially equivalent descriptions of the problem result in different equilibrium prices and thereby different utility levels for the involved parties. If this were the case, the mere re-scaling of the problem, through replicating the bargaining powers of parties on the one hand and correspondingly replicating the available quantities of resources on the other hand, would yield different utility levels for different agents. Such inconsistencies could open the door to manipulations in the description of problems. Replacement consistency is implied by our axiomatic characterization. It limits

[2]That is, with the preference that admits a representation $u(y) = p \cdot y$ for every $y \in \mathbb{R}_+^L$.

the choice of equilibria only in some problems where all agents but one are linear with the same characterizing vector of coefficients.

The following theorem states that any solution that satisfies the axioms matches each dispute with an exchange economy and selects one equilibrium price vector in that economy. Following Arrow and Debreu [1], an equilibrium price vector indeed exists. The solution then assigns as a resolution of the dispute all the equilibrium allocations that are distributions of x, corresponding to the selected price vector (i.e., in case of a good with a zero price the solution will assign a subset of the corresponding equilibrium allocations, composed of those equilibrium allocations that do not generate excess supply).

Theorem 1. *Let φ be a bargaining solution to problems $(x, (\succsim^i)_{i=1}^n, d)$ that satisfy the structural assumption **A0**. The following two statements are equivalent:*

*(i) φ satisfies assumptions **A1-A4**.*

(ii) For any bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$, φ assigns the set of all equilibrium allocations that distribute x, corresponding to one equilibrium price vector in a pure exchange economy where agent i holds preferences $\succsim^i$ and is endowed with $d(i) + \frac{1}{n}\left(x - \sum_{j=1}^{n} d(j)\right)$, $i = 1, \ldots, n$. In addition, φ is replication-consistent and replacement-consistent.

The essence of different disagreement bundles is that agents possess different bargaining powers. Our axioms characterize a solution that takes different bargaining powers into account by translating them into endowments in the respective exchange economy. The endowment of an agent in that economy becomes his or her disagreement bundle plus an equal share of the remainder. These endowments are akin in spirit to the 'Contested Garment Principle' of Aumann and Maschler [2], prescribing equal division of the contested amount, if each party i is taken to claim $d_i + (x - \sum_{j=1}^n d_j)$ of the resources under consideration. Such claims reflect the supposition that disputing parties accept their peers' rights over their default bundles and thus claim only what remains of the resources after deducing others' disagreement bundles.

The theorem implies that an impartial arbitrator is selected for every bargaining problem. This is a linear preference relation whose vector of coefficients is an equilibrium price vector in the corresponding exchange economy. In addition, a solution is in particular individually rational, since agents prefer their assigned bundles over their default ones. As far as we are aware of, this is the first axiomatic characterization of a competitive equilibrium in an exchange economy, that allows agents to start with different endowments.

2.3 Examples

Imagine two business partners who need to decide on the distribution of one acre of land, one million dollars and two tons of wheat seed. The resources to be split are $(1, 1, 2)$, where the coordinates denote acres of land, millions of dollars and tons of wheat seed, respectively. The preference relations of the two partners involved are denoted by $\succsim^1$ and $\succsim^2$. For the sake of the illustration suppose that these preferences belong to the Cobb-Douglas class with parameters $(1/2, 1/3, 1/6)$ for $\succsim^1$ and parameters $(1/4, 1/2, 1/4)$ for $\succsim^2$.

First consider the situation in which if bargaining fails both partners receive nothing. The market bargaining solution for that case suggests that partners be allocated their equilibrium bundles in the exchange economy in which each is endowed with half the resources, namely with $(1/2, 1/2, 1)$. Given the partners' Cobb-Douglas preferences, the market bargaining solution allocates $(2/3, 2/5, 4/5)$ to the first partner and $(1/3, 3/5, 6/5)$ to the second. This will in fact be the solution to every problem that assigns the partners with identical bundles in case bargaining fails.

To understand how a solution depends on parties' bargaining power we compare the above situation to that in which if bargaining fails the first partner ends up with the bundle $(0.25, 0, 0.5)$ and the second with $(0.5, 0.5, 0)$. The market bargaining solution for that problem entails that the partners be allocated their equilibrium allocations in the exchange economy in which the first partner is endowed with $(0.375, 0.25, 1.25)$ and the second partner with $(0.625, 0.75, 0.75)$. Solving the equilibrium for these parameters implies that according to the market bargaining solution the first partner will be allocated the bundle $(6/11, 2/7, 4/7)$ and the second will be allocated $(5/11, 5/7, 10/7)$. As can be seen, the first partner's allocation under the given circumstances is worse than his or her allocation in the first, symmetric case. The reason is that the first partner entered the bargaining situation with less bargaining power than the second partner, thus in an inferior position to his or her own position in the symmetric bargaining situation.

The difference between the partners' allocations in the symmetric versus the asymmetric situations stands in sharp contrast to the results obtained under classic bargaining solutions. To further illustrate the difference, suppose that in the above example the two partners hold identical Cobb-Douglas preferences, characterized by parameters $(1/2, 1/3, 1/6)$. When disagreement bundles are identical (as in the all-zeros disagreement point, for instance) both partners possess the same bargaining power and as in all classic bargaining solutions the market bargaining solution will yield them equal bundles. Now, however, assume that the bargaining situation is such that in case bargaining fails the first partner gets all the land and all the wheat seeds while the second one gets nothing.[3] Any utility-based solution cannot distinguish between the two problems since both partners' default utilities, in both bargaining stories (i.e. the symmetric and the seemingly asymmetric one), are zero. Thus, any utility-based solution will still prescribe an even distribution of resources. Yet, we argue that it is not intuitive to ascribe to both partners the same bargaining power. After all, the partner with the land and wheat invests more in the partnership to begin with. This partner may therefore be unwilling to enter a partnership in which this larger investment is not recognized. In contrast to the utility-based approach, the market bargaining solution will distinguish between the two starting positions, and assign as a resolution the equilibrium allocation in the exchange economy in which the first partner is endowed with $(1, 0.5, 2)$ and the second partner with $(0, 0.5, 0)$. The market resolution to this dispute will therefore consist of allocating one sixth of the resources to the second partner, and five times as much to the first partner.

2.4 An alternative characterization for the equal-rights case

When all agents' disagreement allocations are the same, we are able to offer an alternative, simpler characterization of the market solution. This characterization requires less assumptions on an arbitrating preference, adding instead a basic fairness condition. It employs a weakened definition of bilateral negotiations, whereby it is only required that every agent obtain the same bundle under the solution to the original problem and when separately negotiating with the arbitrator, but no assumption of fairness of the allocation a' in the replaced problem is made. Parties will therefore not object to conducting bilateral negotiations with the assigned arbitrator, as they will obtain the same bundle (or its preference-worth), and moreover, they do not suspect that their peers can manipulate the solution by separately negotiating with the arbitrator. Impartiality in this alternative characterization employs the weakened form of the definition, but other than that, remains unchanged.

Formally, say that a bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$ is symmetric if $d(i) = d(j)$ for every i and j. A binary relation $\succsim^*$ *can bilaterally negotiate* $a \in \varphi(x, (\succsim^i)_{i=1}^n, d)$ if for any agent i there exists an allocation $a' \in \varphi(x, (\succsim^i, \underbrace{\succsim^*, \ldots, \succsim^*}_{n-1 \text{ times}}), d)$ with $a'(i) = a(i)$.

Apart from the weakened version of Impartiality and axioms **A1**, **A2** and **A4** stated above, another axiom is now added to the characterization. It asserts that if two bargaining parties have the exact same preferences, then neither of them finds the other's bundle to be better (see Thomson [25]). Whenever a bargaining problem contains agents with identical preferences (as in the land example in the Introduction), these agents will most likely reject a solution which

[3] These disagreement bundles do not satisfy the condition in A0d, but the same essential point can be made with *almost* all the land, and *almost* all the wheat seeds given to the first partner.

may arbitrarily prefer one of them. Agents that do not have peers with the same preferences would be neither against nor in favor of this assumption.

A5. Equal Treatment of Equals.
Let $(x, (\succsim^i)_{i=1}^n, d)$ be a symmetric bargaining problem and $a \in \varphi(x, (\succsim^i)_{i=1}^n, d)$ an allocation in its solution. If $\succsim^i = \succsim^j$, then $a(j) \sim^i a(i)$.

An alternative characterization of the market solution for symmetric bargaining problems ensues.

Theorem 2. *Let φ be a bargaining solution to symmetric bargaining problems that satisfy the structural assumption **A0**. The following two statements are equivalent:*

*(i) φ satisfies assumptions **A1-A5**, with **A4** modified as explained above.*

(ii) For any symmetric bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$, φ assigns all the equilibrium allocations that distribute x, corresponding to one equilibrium price vector in a pure exchange economy where agents hold preferences $(\succsim^i)_{i=1}^n$ and each of them is endowed with x/n. In addition, φ is replication-consistent and replacement-consistent.

3 Proofs

3.1 Proof of Theorem 1

3.1.1 Proving that (i) implies (ii)

Let φ be a bargaining solution to bargaining problems that conform to the structural assumption **A0**, and suppose that φ satisfies assumptions **A1-A4**.

Step 1: It is first shown that if a binary relation $\succsim^*$ conforms to A0(2) and is unprejudiced, then $\succsim^*$ must be a linear relation, satisfying monotonicity. That is to say, there is a vector of coefficients $m \in \mathbb{R}_+^L$ such that $\succsim^*$ ranks bundles $y \in \mathbb{R}_+^L$ according to the utility function $u_*(y) = m \cdot y$.

Lemma 1. *Suppose an unprejudiced binary relation $\succsim^*$, that conforms to A0(2). Then there is a vector $m \in \mathbb{R}_+^L$ such that $y \succsim^* z$ if and only if $m \cdot y \geq m \cdot z$.*

Proof. It is first proved that each indifference curve of $\succsim^*$ is a hyperplane. Let $t >> 0$ and consider a tangent hyperplane to the indifference curve of $\succsim^*$ that goes through t. Denote its characterizing coefficients by $m \in \mathbb{R}_+^L$ (a non-negative such tangent exists on account of A0(2b)). Note that given that $\succsim^*$ satisfies A0(2) (and thus satisfies in particular convexity) it holds that $t \succsim^* z$ for every $z \in \mathbb{R}^L_+$ for which $m \cdot z = m \cdot t$. Suppose on the contrary that the $\succsim^*$-indifference curve going through t is not a hyperplane, so that there is $y \in \mathbb{R}^L_+$ such that both $y \sim^* t$ and $m \cdot y > m \cdot t$.

Let λ be a number greater than 1. Since $m \cdot y > m \cdot t$, there is $\varepsilon(\lambda) > 0$ such that (i) $\varepsilon(\lambda)\lambda m \cdot t = (\lambda - 1)m \cdot (y - t)$, and (ii) $\varepsilon(\lambda) \to 0$ as $\lambda \to 1$. Set, $z = \lambda(1 + \varepsilon(\lambda))t + (1 - \lambda)y$, which for λ close enough to 1 is in $\mathbb{R}^L_+$. Thus, $m \cdot z = m \cdot t$, which implies $t \succsim^* z$.

Note that $(1 + \varepsilon(\lambda))t$ is on the interval connecting y and z. More precisely, $(1 + \varepsilon(\lambda))t = (1 - \frac{1}{\lambda})y + \frac{1}{\lambda}z$. Let $\lambda = \frac{n}{k}$ be a rational number, with $n > k > 0$ being integers. Letting $\varepsilon = \varepsilon(\frac{n}{k})$ we obtain,

$$(1 + \varepsilon)t = (1 - \frac{k}{n})y + \frac{k}{n}z \ . \tag{1}$$

When $\frac{k}{n}$ is sufficiently close to 1, $z >> 0$ (because $t >> 0$) and $y - \varepsilon t, z - \varepsilon t \geq 0$.

We now construct allocation $b \in \mathcal{A}_n(nt)$ by assigning $b(i) = z - \varepsilon t$ for $i = 1, ..., k$ and $b(i) = y - \varepsilon t$ for $i = k + 1, ..., n$. Allocation b is indeed in $\mathcal{A}_n(nt)$, because by Eq. (1), $k(z - \varepsilon t) + (n - k)(y - \varepsilon t) = nt$. We compare b to the constant allocation $a \in \mathcal{A}_n(nt)$, assigning $a(i) = t$, for every $i = 1, ..., n$.

Recall that $t \sim^* y$ and $t \succsim^* z$. According to A0, $\succsim^*$ is monotonic and due to $t >> 0$, we obtain $a(i) \succ^* b(i)$ for every $i = 1, ..., n$. This is a contradiction to the assumption that $\succsim^*$ is unprejudiced. We conclude (after applying also the continuity of the relation) that $\succsim^*$ generates hyperplane indifference curves.

It remains to show that the hyperplane indifference curves are parallel. Suppose on the contrary two indifference curves which are not parallel. Let $t_1 >> 0$ lie on the lower curve, and let t_2 be the point on the second indifference curve which lies on the ray from the origin that crosses t_1, namely $t_2 = (1 + \alpha)t_1$ for some $\alpha > 0$.

As the two hyperplane indifference curves in question are non-parallel, there is a vector v such that both $t_2 + v$ and $t_2 - v$ are in the hyperplane of t_2, while $t_1 + v$ is above and $t_1 - v$ is below the hyperplane of t_1. In terms of preferences it implies that $z_2 := t_2 + v \sim^* t_2$, while $z_1 := t_1 - v \prec^* t_1$. Denote $t := (t_1 + t_2)/2$. Note that $2t = z_1 + z_2$, implying that (z_1, z_2) is a split of $2t$ into two. Furthermore, $z_2 \sim^* t_2$ but $t_1 \succ^* z_1$. Due to continuity and monotonicity there is $\lambda > 0$ such that $w := z_1 + \lambda t \sim^* t_1$. We thus obtain, $w + z_2 = z_1 + \lambda t + z_2 = (2 + \lambda)t$.

We now construct an allocation $b \in \mathcal{A}_2(2t)$: $b(1) = w - (\lambda/2)t$ and $b(2) = z_2 - (\lambda/2)t$. Note that $b(1)$ is strictly smaller than w and $b(2)$ is strictly smaller than z_2, and therefore $w \succ^* b(1)$ and $z_2 \succ^* b(2)$. By comparing allocation b to the allocation $a \in \mathcal{A}_2(2t)$, composed of $a(1) = t_1$ and $a(2) = t_2$, a contradiction is inflicted upon the assumption that $\succsim^*$ is unprejudiced. This leads us to assert that $\succsim^*$ is characterized by parallel hyperplane indifference curves, namely there is $m \in \mathbb{R}^L_+$ (where nonnegativity of the components of m stems from monotonicity of the relation) such that for every two bundles y and z, $y \succsim^* z$ if and only if $m \cdot y \geq m \cdot z$. ■

Denote a disagreement point by $d = (d(1), \ldots, d(n))$ with $d(i) = (d_1(i), \ldots, d_L(i))$ being the bundle allocated to agent i in case of disagreement. Consider a bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$, satisfying **A0**, and let a denote an allocation in its solution. By Impartiality (**A3**) there exists an unprejudiced relation $\succsim^*$ that conforms to A0(2), which perceives a as fair given d, and can bilaterally negotiate it. According to Lemma 1, $\succsim^*$ is a linear preference. Namely, there is a vector of coefficients $m = (m_1, \ldots, m_L)$ such that for any two bundles y and z, $y \succsim^* z$ if and only if $m \cdot y \geq m \cdot z$. As $\succsim^*$ is monotone, $m \geq 0$.

The linear preference $\succsim^*$ perceives a as fair given d. Namely, there exists a bundle $t \in \mathbb{R}_+^L$ such that $a(i) \sim^* d(i) + t$ for all i. Simple arithmetics yields that $m \cdot t = m \cdot \dfrac{(x - \sum_{j=1}^n d(j))}{n}$. Therefore, $m \cdot a(i) = m \cdot d(i) + m \cdot \dfrac{(x - \sum_{j=1}^n d(j))}{n}$ for every i. Denote $w = m \cdot \dfrac{(x - \sum_{j=1}^n d(j))}{n}$.

Fix an agent i, and consider the replaced bargaining problem $(x, (\succsim^i, \underbrace{\succsim^*, \ldots, \succsim^*}_{n-1 \text{ times}}), d)$, created when all agents $j \neq i$ are replaced by $\succsim^*$, where a copy of $\succsim^*$ that replaces agent j also inherits j's disagreement bundle $d(j)$. According to Impartiality (**A3**), for any allocation a' in the solution to this replaced problem, $a'(i) \sim^i a(i)$, and a' is perceived by $\succsim^*$ as fair given d. Following the above, it holds that $m \cdot a'(j) = m \cdot d(j) + w = m \cdot a(j)$ for all $j \neq i$, hence $a'(j) \sim^* a(j)$ for all $j \neq i$. The Equivalence Principle (**A2**), applied to the replaced problem, implies that a is also in the solution to the replaced problem, and a copy of $\succsim^*$ that replaces an agent $j \neq i$ gains a utility value of $m \cdot d(j) + w$ in the solution to this replaced problem.

Step 2: Let $y \in \mathbb{R}_+^L$ be a bundle that satisfies $x \geq y$ and $m \cdot y - m \cdot d(i) = m \cdot \frac{(x - \sum_{j=1}^n d(j))}{n} = w$. Note that in this case it holds that $m \cdot (x - y) = (n-1)w + \sum_{t \neq i} m \cdot d(t)$. Define:

$$\begin{aligned} \lambda &= \frac{(n-1)w}{(n-1)w + \sum_{t \neq i} m \cdot d(t)} \\ \theta_j &= \frac{m \cdot d(j)}{\sum_{t \neq i} m \cdot d(t)} \ , \quad j \neq i \end{aligned}$$

Consider the distribution of x that assigns y to agent i and assigns $\dfrac{\lambda(x-y)}{n-1} + \theta_j(1-\lambda)(x-y)$ to the copy of $\succsim^*$ that replaces agent $j \neq i$. Multiplying each of these bundles, for $j \neq i$, by m delivers $w + m \cdot d(j)$. A copy of the relation $\succsim^*$ that replaces agent $j \neq i$ is therefore indifferent between the bundle under this allocation and the bundle $a(j)$. The Pareto assumption implies that $a(i) \succsim^i y$. It follows that $a(i) \succsim^i y$ for any bundle y that satisfies $x \geq y$ and $m \cdot y = m \cdot d(i) + w$.

Step 3: Consider the doubled problem, $(2x, (\succsim^1, \succsim^1, \ldots, \succsim^n, \succsim^n), (d, d))$, consisting of allocating twice the resources to a double-sized group of agents in which there are two clones of every preference $\succsim^i$, $i = 1, \ldots, n$, each with a disagreement bundle identical to that in the original problem. According to Replication Invariance (**A4**) the allocation giving $a(i)$ to every i-clone is included in the solution to the doubled problem, and can also be bilaterally negotiated by $\succsim^*$ as the φ-solution to the doubled problem. Denote the doubled allocation by a^d. The conclusion of the previous step is now applied to the doubled problem. It results that for every agent i and any bundle y such that $2x \geq y$ and $m \cdot y = m \cdot d(i) + w$, $a(i) \succsim^i y$.

By iterating the same argument (with $4x, 8x$, and so on) it may be concluded that any agent i prefers $a(i)$ over any bundle y such that $m \cdot y = m \cdot d(i) + w$. In other words, we obtain that for every bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$ and every agent i, any allocation in $\varphi(x, (\succsim^i)_{i=1}^n, d)$ maximizes i's utility among all the bundles whose worth w.r.t. m is the same as the worth of $d(i) + \frac{(x - \sum_{j=1}^n d(j))}{n}$, for some such vector of coefficients $m = (m_1, \ldots, m_L)$. This is precisely the formulation of the constrained optimization problem of a pure exchange economy where each agent i is initially endowed with $d(i) + \dfrac{1}{n}\left(x - \displaystyle\sum_{i=1}^n d(i)\right)$, and the competitive equilibrium price vector is m.

Step 4: We show here that a vector m as above does exist. By **A0** (see Arrow and Debreu [1]) there exists a competitive equilibrium for the above exchange economy. Thus, a vector m exists as desired – it is a prevailing competitive price system. The allocation a is an equilibrium allocation corresponding to the equilibrium prices given by $(m_1, \ldots, m_L)$, under endowment of $d(i) + \frac{(x - \sum_{j=1}^n d(j))}{n}$ for agent i, for all i. Following the Equivalence Principle (**A2**), all the equilibrium allocations that correspond to the prices $(m_1, \ldots, m_L)$ are included in the solution. The next claim shows that these are the only allocations included in the solution.

Claim 1. *Suppose that $a \in \varphi(x, (\succsim^i)_{i=1}^n, d)$ is an equilibrium allocation corresponding to an equilibrium price vector m, for endowments $d(i) + \dfrac{1}{n}\left(x - \displaystyle\sum_{i=1}^n d(i)\right)$ per agent i. Then for any $a' \in \varphi(x, (\succsim^i)_{i=1}^n, d)$, a' is also an equilibrium allocation in the same exchange economy, that corresponds to the same price vector m.*

Proof. Let $a = (a(i))_{i=1}^n$ and $a' = (a'(i))_{i=1}^n$ be two allocations in $\varphi(x, (\succsim^i)_{i=1}^n, d)$, and suppose that a is an equilibrium allocation in an exchange economy with the above endowments, corresponding to the price vector m. The Equivalence Principle (**A2**) entails that all the agents are indifferent between the two allocations: $a(i) \sim^i a'(i)$ for every i.

Since a is an equilibrium allocation, $a(i)$ - the share of i - optimizes i's utility subject to i's budget constraint. Suppose that there is an agent j whose allocation $a'(j)$ satisfies $m \cdot a'(j) - m \cdot d(j) > m \cdot a(j) - m \cdot d(j)$. Following the fairness of allocations assigned by the solution, there must be another agent i whose allocation $a'(i)$ satisfies $m \cdot a'(i) - m \cdot d(i) < m \cdot a(i) - m \cdot d(i)$. This would make $a'(i)$ an allocation that does not exceed i's budget constraint in the above economy, yet is equivalent to $a(i)$ (by the Equivalence Principle, as explained above). By **A0**, all the agents' preferences are monotonic. Thus agent i could improve on $a(i)$, within the limit of his or her budget constraint, by adding a strictly positive bundle to $a'(i)$. It would follow that $a(i)$ is not optimal for i under i's budget constraint, contradicting the assumption that $a(i)$ is an equilibrium allocation as specified above. Therefore, we must conclude that $m \cdot a'(i) - m \cdot d(i) = m \cdot a(i) - m \cdot d(i)$ for every i. It follows that $a'(i)$ is optimal for every i under the budget constraint corresponding to m, making the entire allocation a' an equilibrium allocation in the exchange economy with endowments $d(i) + \frac{1}{n}\left(x - \sum_{i=1}^{n} d(i)\right)$, $i = 1, \ldots, n$, under the price vector m. ■

We conclude that, given a bargaining problem, a solution φ to this problem equals the set of all the equilibrium allocations in the exchange economy where each agent i is endowed with $d(i) + \frac{1}{n}\left(x - \sum_{i=1}^{n} d(i)\right)$, corresponding to some vector of equilibrium prices m .

Step 5: The last step in the proof consists of asserting that there exists a selection of equilibria across different problems that is well defined, and satisfies both replication consistency and replacement consistency. For that matter, a bargaining problem is a *replicated problem* if it is a k-replication (for some $k \in \mathbb{N}$) of another bargaining problem. And it is a *replaced problem* if there are $n-1$ identical, linear preferences out of its n involved preferences, where the vector of coefficients characterizing these linear preference is an equilibrium price vector in the exchange economy corresponding to this problem. Note that if agents in a problem are replaced by a linear preference characterized by an equilibrium price vector for that problem, then the same equilibrium price vector prevails in the resulting replaced problem.

For any bargaining problem which is not replicated nor replaced, the selection of an equilibrium price vector is unconstrained. Any equilibrium price vector may be selected, and all of its corresponding equilibrium allocations are assigned as a resolution. Given that selection, Replication Invariance (**A4**) dictates the selection in replicated problems. Impartiality (**A3**), through the requirement that the equilibrium price vector as a linear agent can bilaterally negotiate the solution, dictates the selection in replaced problems.

Finally, the above choices are well-defined as any single replicated or replaced problem cannot be obtained from two different problems with different equilibria selections. Replicated problems can always be traced to the maximal number of replications, and by Replication Invariance (**A4**), the same goes for the choice in any other replicated problem along the way. With regard to replaced problems, whenever the number of agents is at least three a replaced problem can only originate from one root problem. A problem may arise only in problems involving two agents with differing linear preferences, in case both characterizing coefficients of these preferences constitute equilibrium price vectors for the implied exchange economy. However it is well known that with linear preferences (under our structural assumption **A0**) equilibrium is unique. Therefore, such a situation cannot obtain and the selection described above is well defined. Moreover, it satisfies both replication consistency and replacement consistency.

3.1.2 Proving that (ii) implies (i)

Suppose that (ii) of the theorem holds. Assumption **A1** readily obtains since all equilibrium allocations are Pareto optimal.

In order to prove **A2**, observe first that all the allocations corresponding to the same vector of equilibrium prices are indifferent for all agents. For the other part of **A2**, suppose that $a = (a(i))_{i=1}^n$ and $a' = (a'(i))_{i=1}^n$ are two allocations between which all agents are indifferent: $a(i) \sim^i a'(i)$ for every i. Assume that a is an equilibrium allocation corresponding to an exchange economy with endowments $d(i) + \frac{1}{n}\left(x - \sum_{i=1}^{n} d(i)\right)$, $i = 1, \ldots, n$. Consequently, according to the proof of Claim 1, a' is an equilibrium allocation for the same prices, and thus either both are assigned by a solution or none is.

Impartiality (**A3**) is implied since for any equilibrium allocation the corresponding vector of equilibrium prices can be extended to a linear preference relation, which is unprejudiced as a simple implication of its definition, and assigns the same value to every $a(i) - d(i)$, for each equilibrium allocation a and every agent i. The same equilibrium prices prevail in the original exchange economy and in the exchange economy constructed by any one of the original agents and $n - 1$ copies of the linear prices-agent, when each replacing agent is endowed with the disagreement bundle of the replaced agent. By replacement consistency the selected equilibria both in the original problem and in any replaced problem are the same. Thus, the allocations that are assigned by a solution in the original problem are also assigned by this solution in the replaced problem.

Part (a) of Replication Invariance (**A4**), which states that replications of allocations in a solution are included in the solution to the replicated problem, follows from the assumption that a solution is replication-consistent. This restricts a solution to choose the same market equilibrium for both the replicated and the original problems.

To prove part (b) of Replication Invariance, suppose a bargaining problem $(x, (\succsim^i)_{i=1}^n, d)$ and an equilibrium allocation a in its solution. Denote by $p = (p_1, \ldots, p_L)$ the vector of equilibrium prices corresponding to a, and normalize it by $p \cdot x = n$. For every agent i it holds that $p \cdot a(i) = p \cdot d(i) + p \cdot \dfrac{x - \sum_k d(k)}{n}$.

Let $\succsim^*$ be an unprejudiced binary relation that conforms to A0(2), which perceives a as fair given d and can bilaterally negotiate it as a φ-solution to the problem. It should be proved that for any $k \in \mathbb{N}$, $\succsim^*$ can bilaterally negotiate the allocation that is a k-replication of a as the φ-solution to the bargaining problem that is a k-replication of the problem under consideration. Once again, Lemma 1 implies that the relation $\succsim^*$ under consideration is linear. Denote its coefficients by $m = (m_1, \ldots, m_L)$, normalized so that $m \cdot x = n$. Fairness of a given d according to $\succsim^*$ means that $(m \cdot a(i) - m \cdot d(i))$ is constant across i. Therefore, by summing over i, $m \cdot a(i) = m \cdot d(i) + m \cdot \dfrac{x - \sum_k d(k)}{n}$.

Given a good ℓ, let i be an agent such that $a_\ell(i) > 0$. Now examine the replaced problem in which $\succsim^*$ replaces all agents but i (along with their disagreement bundles). Under (b) of Replication Invariance it is assumed that a can bilaterally be negotiated by $\succsim^*$. Thus, following the Equivalence Principle (as shown above in the proof of the other direction) a is included in the solution to this replaced problem. According to assumption **A0(2)** (and specifically its part (e)), since agent i consumes the same bundle $a(i)$ in this replaced problem, and as this agent is still subject to the same budget constraint, the fact that $a_\ell(i) > 0$ implies that $p^i_\ell = p_\ell$. Denote the vector of equilibrium prices in this replaced problem by p^i, where p^i is again normalized so that $p^i \cdot x = n$. A solution assigns equilibrium allocations that correspond to endowments of $d(j) + \dfrac{(x - \sum_{k=1}^n d(k))}{n}$ per agent j. Therefore, here as well it holds that $p^i \cdot a(j) = p^i \cdot d(j) + p^i \cdot \dfrac{x - \sum_k d(k)}{n}$ for every j.

For $j \neq i$, linearity of $\succsim^*$ that replaces j implies that whenever p^i_r is zero then so is m_r. Otherwise, there exists a number t such that for any good r with a non-zero price, for which $a_r(j) > 0$, it holds that $m_r/p^i_r = t$, and for any good r for which $a_r(j) = 0$, $m_r/p^i_r \leq t$. Consequently, $m \cdot a(j) = tp^i \cdot a(j)$ for $j \neq i$. Combining this with the above equalities that pertain to m and to the equilibrium price vectors, it follows that for $j \neq i$, $m\left(d(j) + \dfrac{(x - \sum_{k=1}^n d(k))}{n}\right) = m \cdot a(j) = tp^i \cdot a(j) = tp^i\left(d(j) + \dfrac{(x - \sum_{k=1}^n d(k))}{n}\right)$. Namely, $(m - tp^i)\left(d(j) + \dfrac{(x - \sum_{k=1}^n d(k))}{n}\right) = 0$. As $m - tp^i \leq 0$ and $d(j) + \frac{(x - \sum_{k=1}^n d(k))}{n} >> 0$ it must be that $m_r = tp^i_r$ for every good r. The identical normalization of m and p^i delivers $t = 1$. That is, $m_r = p^i_r$ for every r, and specifically, $m_\ell = p^i_\ell = p_\ell$. Repeating the same arguments for every ℓ yields that $m_\ell = p_\ell$ for every good ℓ. In summary, whenever an equilibrium allocation in the exchange economy corresponding to a bargaining problem is perceived as fair and can be bilaterally negotiated by an unprejudiced relation $\succsim^*$ that conforms to A0(2), then $\succsim^*$ is a linear relation characterized by the vector of coefficients p, p being the vector of equilibrium prices corresponding to that allocation.

Replication consistency and replacement consistency guarantee that the same equilibrium price vector is chosen for $(x, (\succsim^i)_{i=1}^n, d)$ and for any replaced problem created from its k-replication by replacing all agents but one with $\succsim^*$. Therefore, under the chosen equilibrium prices for these replaced problems, $\succsim^*$ is indifferent between all bundles that exhaust the budget constraint. Since, as already established, any allocation contained in the solution to the original problem is also contained in the solution to its k-replication, and as any such allocation exhausts the budget constraint, any allocation in the solution to the original problem is contained in the solution to any replaced problem created from the replicated problem. Namely, $\succsim^*$ can bilaterally negotiate a^k as the φ-solution to the k-replicated problem.

3.2 Proof of Theorem 2

Note first that for symmetric problems, fairness of an allocation according to a linear relation $\succsim^*$ characterized by a vector of coefficients $m = (m_1, \ldots, m_L)$, means that $m \cdot a(i)$ is constant across all agents $i = 1, \ldots, n$.

The only part that differs from the proof of Theorem 1 is in step 1, where it is shown that if an allocation a is in the solution to the original problem, then it is also in the solution to the replaced problem, in which $\succsim^*$ replaces all agents but one. To prove that part, recall that according to the weakened definition of bilateral negotiations, there exists an allocation a' in the solution to this replaced problem such that $a'(i) = a(i)$. In addition, the assumption of Equal Treatment of Equals (**A5**) implies that the bundles allocated under a' to the copies of $\succsim^*$ are $\succsim^*$-indifferent, hence for every $j \neq i$, $m \cdot a'(j) = m \cdot (x - a'(i))/(n-1) = m \cdot (x - a(i))/(n-1) = m \cdot a(j)$. It follows that $a'(j) \sim^* a(j)$ for each $j \neq i$. The Equivalence Principle (**A2**), applied to the replaced problem, delivers that $a \in \varphi(x, (\succsim^i, \underbrace{\succsim^*, \ldots, \succsim^*}_{n-1 \text{ times}}), d)$. Other than that, the proof is the same as that of Theorem 1.

In the other direction, the proof that **A1-A4** hold is the same as that proof for Theorem 1, and **A5** is implied by the definition of competitive equilibrium and the identical budget constraint for all agents.

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