Maxmin expected utility as resulting from information on others' preferences *

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— *Extremely Preliminary Draft* —

Abstract

This paper extends the Maxmin Expected Utility model of Gilboa and Schmeidler (1989) by accommodating information as part of the setup. The decision maker in the paper is assumed to have access to preferences of other individuals, where these preferences abide by Savage's Subjective Expected Utility theory. These preferences constitute the information that is available to the decision maker. Disagreement on likelihood rankings within the information gives rise to ambiguity for the decision maker. Given the information and based on the ambiguity it induces, the paper offers an axiomatic characterization of a decision maker who follows a Maxmin Expected Utility (MEU) decision rule, entertaining a set of priors instead of a single one. Moreover, the derived set of priors is linked to the primitive information by showing that it never exceeds the convex hull of the underlying subjective prior probabilities. Thus no ambiguity prevails when probability is unanimous according to the available information. The development is performed in an environment *á*-la Savage.

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1 Introduction

Gilboa and Schmeidler (1989) presented an axiomatic model for decision making under uncertainty. Their model generalizes the expected utility paradigm by allowing the decision maker to entertain a set of priors over the states of nature rather than a single prior. The motivation for relaxing the single prior assumption is the understanding that when circumstances are uncertain a decision maker may not be willing to commit to an accurate prior. Resorting instead to a set of priors allows for an expression of ambiguity regarding the likelihoods of the states of nature. The decision maker in the Gilboa and Schmeidler model is assumed to respond aversely to that ambiguity, evaluating each alternative by its minimum expected utility over the set of priors. The model is thus termed Maxmin Expected Utility (MEU).

In the original paper of Gilboa and Schmeidler the source for the ambiguity perceived by the decision maker is not modeled. The set of priors employed by the decision maker is derived endogenously from preference, not tied to any primitive of the model. It thus remains an open question what generates this ambiguity, in which situations will it prevail, and moreover how does it relate to parameters of the decision problem. In this paper we provide one possible answer to these questions, describing situations in which the information available to the decision maker gives rise to ambiguity, namely to uncertainty with regard to the likelihoods of the states of nature. In those situations we tie the set of priors entertained by the decision maker to the primitive of the model that describes the information.

The decision maker in our model is assumed to be faced with individuals holding preferences over acts, where these preferences admit a Subjective Expected Utility (SEU) representation due to Savage (1954). This collection of preferences, given as a primitive of the model, is taken to be the only information available to the decision maker in forming his or her belief over the states of nature. Multiplicity of beliefs that is encapsulated in the information generates ambiguity regarding the 'correct' prior to hold. Given the individual preferences and based on the ambiguity they induce, the paper offers an axiomatic characterization of a decision maker that abides by Gilboa and Schmeidler's Maxmin Expected Utility (MEU) decision model (1989). Moreover, the derived set of priors is linked to the primitive information by showing that it never exceeds the convex hull of the given preferences' subjective prior probabilities.

Our goal in this paper is to further motivate the Maxmin Expected Utility rule by demonstrating how it naturally emerges when one becomes informed of other people's preferences. The point is emphasized by illustrating that even when the information contains only preferences that do not accommodate ambiguity, this nevertheless arises as a result of contradictory underlying beliefs (when the original information already accounts for ambiguity it will most surely generate ambiguity for the decision maker).

Information of the sort we consider may arise when the decision maker has access to preferences of members in his or her society, for instance. Having formed a subjective prior probability, the decision maker becomes aware of others' preferences, and responds by relaxing his or her accurate belief to account for their beliefs as well (In that case the ambiguity-neutral, initial preference of the decision maker will be included as part of the information). Information in the form of preferences may also be obtained by observing individuals who are experts in the decision problem under consideration. The experts are knowledgeable enough not to perceive ambiguity, thus following the SEU paradigm, admitting an accurate prior over the states of nature. Yet the experts do not necessarily agree in totality on the probabilities of events. The decision maker employs their (possibly different) priors to formulate a multiple-priors belief. Importantly, the experts need not share the utility function of the decision maker. Each of them holds personal preferences, characterized by a personal utility function, which may vary across experts. Still the decision maker relies on their likelihood judgements, regardless of their utilities (We only require that there be one pair of consequences which everybody rank in the same manner).

The basic conception that the axioms convey is that the ambiguity perceived by the decision maker is only the result of conflicting likelihood assessments in the given information. Once all those preferences agree on underlying probabilities no ambiguity arises. Accordingly, the decision maker regards as unambiguous every act that generates the same distribution over outcomes according to all the available information. This is reflected in an axiom termed Unambiguous Independence whereby the decision maker regards each such act as if it were a lottery with known probabilities, keeping rankings of (any) acts unchanged under mixtures with such 'lotteries'. On the other hand when ranking ambiguous acts, for which the information induces conflicting probabilities, a tendency of the decision maker towards hedging is imposed through an ambiguity aversion axiom.

The entire work is performed in a Savage-like framework, meaning that the setup is purely subjective with no exogenous probabilities assumed. Furthermore, richness is obtained through a non-atomicity axiom rather than supposed as part of the framework, so that the outcomes set is general and can even consist of a single pair of consequences. To the best of our knowledge, this is the first axiomatic characterization of the MEU rule in this type of setup. Previous purely subjective derivations of the MEU rule relied on an a-priori rich set of consequences, in the form of a connected topological space that is assumed as part of the setup. These are Casadesus-Masanell et al. (2000), Ghirardato et al. (2003), and Alon and Schmeidler (2014).

Gajdos, Hayashi, Tallon and Vergnaud (2008) suggested (in an Anscombe Aumann, 1963, setup, as rephrased by Fishburn, 1970) another decision model in which information is defined as part of the framework and where the decision maker evaluates acts according to a Maxmin rule. In their model information takes the form of a set of priors, interpreted as an exogenously given set containing all possible priors in the situation under consideration. The primitive preference which is the subject matter of the study is defined over pairs of act and priors set, so that the decision maker compares one act with one set of possible priors to another act with another set of possible priors. The information itself is therefore subject to preferences. Embedding the possible priors within the decision maker's preferences allows these authors to characterize different approaches towards precision of available information. On the other hand, it makes the setup less standard, and complies the decision maker to form preferences over more complex objects. To compare, in the current paper the decision maker's preferences are defined over standard Savage acts, and information is fixed throughout the decision problem, given by preferences of individuals. Admittedly, it is then impossible to compare, for a single decision maker, his or her preferences over the available information itself.

In a companion paper Alon and Gayer (2015) characterize a utilitarian social preference, assuming SEU preferences for individuals and an MEU preference for society. A central axiom in the aggregation of individual preferences that takes place there is the axiom we employ here, which restricts the type of ambiguity to be only inflicted by conflicts. Consequently the dependence between the social set of priors and the individual priors takes the same form as in the current paper. The social planner in that model may be thought of as having access to the preferences of the members of society, forming a belief on that basis.

2 Setup and a basic assumption

Let S denote a nonempty set of *states of nature* that is endowed with a sigma algebra Σ of *events*. The set of possible *outcomes* is X and the set of *acts* is $\mathcal{F} = \{f \in X^S \mid f$ obtains finitely many values and is measurable w.r.t. $\Sigma\}$, also known as simple acts. For acts f and g and an event E, fEg stands for the act which assigns the outcome $f(s)$ if $s \in E$ and $g(s)$ otherwise. The decision maker's preferences are modeled by a binary relation over acts, $\succsim$, with asymmetric and symmetric components $\succ$ and $\sim$. An event E is null if $fEh \sim gEh$ for any acts f, g, h, otherwise the event is non-null.

The information given to the decision maker in our model consists of a set of preferences, denoted $\{\succsim^i\}_{i=1}^n$, which are binary relations over $\mathcal{F}$, with $\succ^i$ and $\sim^i$ that denote corresponding asymmetric and symmetric components. Throughout the paper it is assumed that the given individuals' preferences satisfy the axioms of the Subjective Expected Utility model of Savage (1954), therefore each $\succsim^i$ is represented by a functional

$$EU_i(f) = \mathbb{E}_{P_i}(u_i \cdot f) \ ,$$

for a cardinal utility function[1] over outcomes, u_i, and a unique non-atomic subjective probability measure over events[2], P_i, which is further assumed to be sigma-additive (this may be characterized axiomatically by requiring that each relation $\succsim^i$ satisfy a monotone continuity condition, see Villegas, 1964).

A weak notion of conformity is required for the aggregation, whereby the decision maker, as well as the preferences on which the decision maker relies, agree upon the strict ranking of some pair of consequences, as stipulated next.

Minimal Agreement. There are outcomes x^* and x_* such that $x^* \succ x_*$ and $x^* \succ^i x_*$ for all i.

The notation x^*, x_* is henceforth reserved for a specific pair of outcomes that satisfy the Minimal Agreement assumption, and all the utilities u_i are normalized so as to satisfy $u_i(x^*) = 1$ and $u_i(x_*) = 0$.

[1] That is, the utility is unique up to an increasing linear transformation.

[2] A probability measure p is non-atomic if, for any event E with $p(E) > 0$ and any $\lambda \in (0, 1)$, there exists an event $F \subset E$ such that $p(F) = \lambda p(E)$.

3 Axioms

The axioms formulated in this section are shown to imply a representation of the form: for any two acts f and g,

$$f \succsim g \iff \min_{p \in \mathcal{C}} \mathbb{E}_p(u \cdot f) \geq \min_{p \in \mathcal{C}} \mathbb{E}_p(u \cdot f) \ ,$$

for a cardinal utility function u and a unique non-empty, convex and closed set $\mathcal{C}$ of sigma-additive probabilities.[3]

In order to characterize the type of ambiguity that prevails in the situations that are considered in this paper, a definition is required. The definition identifies those cases where there is agreement on the probabilities of events, and corresponding acts that induce the same distribution over outcomes according to all the preferences within the available information. Those acts are employed in the axioms to characterize the situations in which decision maker perceives ambiguity.

Definition 1. *A partition $\{E_1, \ldots, E_m\}$ is an **unambiguous partition** if for every individual i, $x^* E_k x_* \sim^i x^* E_\ell x_*$, for all k and ℓ.*

*An **unambiguous act** is an act which is measurable w.r.t. an unambiguous partition.*

In terms of probabilities, an unambiguous partition $\{E_1, \ldots, E_m\}$ satisfies $P_i(E_k) = \frac{1}{m}$ for every event E_k and every probability P_i. Technically speaking, Lyapunov's theorem guarantees that such partitions exist for every $m \in \mathbb{N}$. An unambiguous act satisfies $P_i(f = x) = P_j(f = x)$ for every outcome x and every pair of probabilities P_i and P_j, namely it generates the same distribution over outcomes according to all the preferences given as information.

The first axiom asserts that whenever the information supports only one direction of likelihood ranking of events, the decision maker will adopt this ranking. It thus limits the ambiguity in our model to emerge only in cases where conflicts over likelihood assessments obtain in the information. Likelihood rankings are elicited from preferences over bets, implemented by the two identically-ranked outcomes, x^* and x_*.

[3] We take the set $\mathcal{C}$ to be closed in the weak* topology over the space of finitely additive set functions. Equivalently for this topology, it means that the set $\mathcal{C}$ is closed under event-wise convergence: if there exists a generalized sequence μ_b in $\mathcal{C}$ such that $\mu_b(A)$ converges to $\mu(A)$ for every event $A \in \Sigma$, then $\mu \in \mathcal{C}$.

No Added Ambiguity. For two events E and F, if $x^*Ex_* \succsim^i x^*Fx_*$ for all i, then $x^*Ex_* \succsim x^*Fx_*$.

No Added Ambiguity describes one sense in which the ambiguity perceived by the decision maker is only the result of conflicts in the information. The next assumption expresses another sense of this constrained ambiguity. Now each of the two acts involved in the axiom may generate different distributions over outcomes across the information, however each of the preferences within the information separately believes that the two acts induce the same distribution over outcomes. The next axiom requires that in that case the two acts will be perceived as indifferent by the decision maker. The resulting condition is a kind of likelihood-sophistication condition for acts which are indifferent in an unambiguous manner: across the preferences within the information the two acts involved are considered equivalent on account of them returning the same outcomes on equally-probable events. The two acts are thus unambiguously equivalent, supported by the information. The condition states that this kind of unambiguous equivalence will be translated by the decision maker into indifference.

Unambiguous Likelihood Sophistication. Let $f = [x_1, E_1; \ldots; x_m, E_m]$ and $g = [x_1, F_1; \ldots; x_m, F_m]$ be two acts. If for every k and every i, $x^*E_kx_* \sim^i x^*F_kx_*$, then $f \sim g$.

A standard weak order assumption is stated next.

S1. Weak Order. For any two acts f and g, $f \succsim g$ or $g \succsim f$ (Completeness). If f, g, h are acts such that $f \succsim g$ and $g \succsim h$ then $f \succsim h$ (Transitivity).

As will become evident, events on which there is an agreed-upon probability play a central role in the characterization. These are recognized using unambiguous partitions and unambiguous acts, as well as the conditional versions thereof, as next defined.

Definition 2. *Given an event E, an **unambiguous partition of** E is a partition $\{E_1, \ldots, E_m\}$ of E such that for every i, $x^*E_kx_* \sim^i x^*E_\ell x_*$, for all k and ℓ.*

*An act is **unambiguous on** E if its part on E is measurable w.r.t. an unambiguous partition of E.*

The preferences given within the information may not agree on the likelihood of an event E but at the same time agree on the likelihood of a sub-event relative to E. In terms of the probabilities, the probability $P_i(E)$ may differ across the information, but for any event E_k in an unambiguous partition of E the conditional probability of E_k given E is the same, so that $P_i(E_k) = \frac{1}{m}P_i(E)$ for each event E_k and every probability P_i. A special case is an unambiguous partition of S, which is exactly the notion of an unambiguous partition that was defined in the previous section.

The next axiom describes the type of monotonicity required for the model. It states that the ranking of two acts that differ only in that they return different unambiguous acts on some event, is determined by the comparison between those unambiguous acts.

S2. Extended Monotonicity. Let $L = [x_1, E_1; \ldots; x_m, E_m]$ and $Q = [y_1, E_1; \ldots; y_m, E_m]$ be two unambiguous acts. Then for any act h and any event E such that $\{E_1 \cap E, \ldots, E_m \cap E\}$ is an unambiguous partition of E, $L \succsim Q$ implies $LEh \succsim QEh$.

The axiom incorporates the rationale of monotonicity and the Sure-Thing principle. The Sure-Thing principle is expressed in that the comparison between the acts LEh and QEh is independent of the act h that is returned outside of E. Monotonicity is reflected in the fact that the ranking of those acts is determined by the stand-alone (unambiguous) comparison between L and Q.

The following two axioms are continuity assumptions. The first is the continuity assumption of Savage's SEU model (his P6), and the second is a monotone continuity condition, known to derive sigma-additive measures in different models (see e.g. Villegas, 1964). In the present model the Monotone Continuity condition implies that all probabilities in the decision maker's set of priors are sigma-additive.

S3. Small Event Continuity. If f, g are acts such that $f \succ g$ and x is an outcome then there exists a finite partition $\{E_1, \ldots, E_n\}$ of S such that $xE_jf \succ g$ and $f \succ xE_jg$ for all events E_j.

S4. Monotone Continuity. Let A_n be a sequence of events such that $A_n \uparrow S$. Then for every non-null event E there exists an index n_0, such that for every $n > n_0$, $x^*A_nx_* \succ x^*(S \setminus E)x_*$.

The last pair of axioms are the heart of the MEU model, describing the ambiguity aversion and independence types that it encompasses. These axioms are stated using half-mixtures of acts, that are now defined.

Definition 3. *A* ***half-mixture of two acts*** $f = [x_1, E_1; \ldots; x_m, E_m]$ *and* $g = [y_1, E_1; \ldots; y_m, E_m]$ *is an act* $h = [x_1, G_1; y_1, E_1 \setminus G_1; \ldots; x_m, G_m; y_m, E_m \setminus G_m]$*, such that for every* k*,* $\{G_k, E_k \setminus G_k\}$ *is an unambiguous partitions of* E_k*.*

The mixture operation consists of dividing each event in the underlying partition into two sub-events which are equally likely according to all the probabilities P_i. A half-mixture of f and g thus assigns the outcome returned by f to an unambiguously half-event of E_k and the outcome returned by g to the remaining unambiguously half-event, thus generating an unambiguous half-mixture of f and g.

The statement of the last two axioms does not require the existence of half-mixtures, but only presents conditions in case they exist. Nevertheless, such mixtures are always available due to Lyapunov's Theorem (this point is discussed and employed in the proof). Typically there is more than one half-mixture of two given acts. However any two such half-mixtures assign the same outcomes to events which are equally likely according to all the probabilities P_i. Thus Unambiguous Likelihood Sophistication renders the decision maker indifferent between the two half-mixtures. As a result the choice of a representative act from the set of half-mixtures of f and g is inconsequential.

S5. Ambiguity Aversion. For any two acts f and g, if h is a half-mixture of f and g, then $f \succsim g$ implies $h \succsim g$.

S6. Unambiguous Independence. Let f and g be two acts and L an unambiguous act. If $h_{f,L}$ is a half-mixture of f and L and $h_{g,L}$ is a half-mixture of g and L, then

$$f \succsim g \iff h_{f,L} \succsim h_{g,L} \quad .$$

These two axioms correspond to those that characterize an MEU decision maker in the original Gilboa and Schmeidler (1989) model, with unambiguous acts in the current model replacing constant acts in the Gilboa and Schmeidler framework. The idea is that the source of ambiguity for the decision maker is the differing likelihood

assessments encapsulated in the available information. When there is unanimity within the information regarding the probabilities of outcomes induced by an act, the decision maker perceives these probabilities as known so that the act presents no ambiguity. Unambiguous acts are therefore treated by the decision maker just as if they were lotteries with exogenous probabilities.

Unambiguous Independence, like Certainty Independence in the Gilboa and Schmeidler (1989) characterization, states that the ranking of acts does not change when they are mixed with an unambiguous act, as the latter act exhibits no ambiguity. Ambiguity Aversion is akin to Uncertainty Aversion that was introduced in Schmeidler (1989), and expresses the decision maker's preference for hedging. The decision maker, by choosing an unambiguous half-mixture of two acts, is able to reduce his or her exposure to uncertainty by offsetting the potential losses and gains.

One last definition is required before stating our main result.

Definition 4. *An event E is a* unanimous half event *if $x^*Ex_* \sim^i x_*Ex^*$ for every i.*

We can now state our MEU characterization theorem, establishing the equivalence between the axioms stated and the MEU decision rule, when information is given as part of the setup in the form of preferences.

Theorem 1. *Let $\succsim$ be a binary relation over $\mathcal{F}$, and $(\succsim^i)_{i=1}^n$ preferences that conform to the SEU model, with utilities $\{u_i\}_{i=1}^n$ and non-atomic, sigma-additive probabilities $\{P_i\}_{i=1}^n$. Suppose that Minimal Agreement is satisfied. Then the following two statements are equivalent:*

(i) $\succsim$ satisfies No Added Ambiguity, Unambiguous Likelihood Sophistication, and assumptions S1-S6.

(ii) There exists a cardinal utility function u and a unique nonempty, closed and convex set $\mathcal{C}$ of sigma-additive prior probabilities[4], such that the relation $\succsim$ is represented by the functional:

$$MEU(f) = \min_{p \in \mathcal{C}} \mathbb{E}_p(u \cdot f) \quad , \quad \forall f \in \mathcal{F} \ ,$$

[4]We take the set $\mathcal{C}$ to be closed in the weak* topology over the space of finitely additive set functions. Equivalently for this topology, it means that the set $\mathcal{C}$ is closed under event-wise convergence: if there exists a generalized sequence μ_b in $\mathcal{C}$ such that $\mu_b(A)$ converges to $\mu(A)$ for every event $A \in \Sigma$, then $\mu \in \mathcal{C}$.

where for some unanimous half-event E, $MEU(x^*Ex_*) = \frac{1}{2}u(x^*)+\frac{1}{2}u(x_*)$. Furthermore, $\mathcal{C}$ is contained in the convex hull of $\{P_i\}_{i=1}^n$.

Supplementing the assumptions above with a condition that appears in Alon and Gayer (2015; Called Social Ambiguity Avoidance there. The social preference $\succsim^0$ should be replaced with $\succsim$ for the result) yields that the decision maker's set of priors equals the convex hull of the probabilities $\{P_i\}_{i=1}^n$, hence the decision maker evaluates acts according to their minimum expected utility over the set of priors that correspond to the information.

4 Proof

4.1 The axioms are sufficient: (i) implies (ii)

Let the preferences $\{\succsim^i\}_{i=1}^n$ be represented by SEU functionals, with utilities $\{u_i\}_{i=1}^n$ and non-atomic, sigma-additive probabilities $\{P_i\}_{i=1}^n$. By the assumption of Minimal Agreement all the utilities can be calibrated to satisfy $u_i(x_*) = 0$ and $u_i(x^*) = 1$. We begin by defining mixtures of acts, based on the probabilities P_i.

Definition 5. *Let $f = [x_1, E_1; \ldots; x_m, E_m]$ and $g = [y_1, E_1; \ldots; y_m, E_m]$ be two acts, such that $\{E_1, \ldots, E_m\}$ is a partition with respect to which both acts are measurable. For $0 \leq \alpha \leq 1$, an $\alpha : (1-\alpha)$ mixture of f and g is an act $[x_1, G_1; y_1, E_1 \backslash G_1; \ldots; x_m, G_m; y_m, E_m \backslash G_m]$ for events G_k that satisfy $G_k \subseteq E_k$ and $P_i(G_k) = \alpha P_i(E_k)$ for every P_i.*

By Lyapunov's theorem such events G_k exist, thus the defined mixtures exist. Moreover, if

$$[x_1, G_1; y_1, E_1 \setminus G_1; \ldots; x_m, G_m; y_m, E_m \setminus G_m] \ , \quad \text{and}$$
$$[x_1, H_1; y_1, E_1 \setminus H_1; \ldots; x_m, H_m; y_m, E_m \setminus H_m]$$

are two $\alpha : (1-\alpha)$ mixtures of f and g, then by definition $P_i(G_k) = P_i(H_k)$ and $P_i(E_k \setminus G_k) = P_i(E_k \setminus H_k)$ for every i and every k, hence by Unambiguous Likelihood Sophistication the decision maker is indifferent between them. A representative act of this indifference class is denoted by $\alpha f \oplus (1-\alpha)g$.

Consider the set of all events that have an agreed-upon probability across all the priors P_i (not necessarily rational). This set of events, denote it by $\mathcal{E}$, is precisely the

sigma-algebra generated by all the unambiguous partitions of S. All the probabilities P_i identify on all the unambiguous partitions. All these probabilities are assumed to be sigma-additive, thus they identify on $\mathcal{E}$ (on account of their continuity). Denote the restriction of some (hence all) beliefs P_i to $\mathcal{E}$ by π. By Lyapunov's theorem again, for any real number $\rho \in [0,1]$ there exists an event $E \in \mathcal{E}$ such that $\pi(E) = \rho$. The set of $\mathcal{E}$-measurable acts is closed under the mixtures defined above, thus these acts are henceforth referred to as *lotteries*. The first stage of the proof is to show that the decision maker's preference over lotteries is represented by a vNM utility function.

Claim 1. *$\succsim$ over lotteries is represented by a vNM utility function.*

Proof. First note that if $L = [x_1, E_1; \ldots; x_m, E_m]$ and $Q = [x_1, G_1; \ldots; x_m, G_m]$ are two lotteries, namely acts measurable w.r.t. $\mathcal{E}$, that satisfy $\pi(E_k) = \pi(G_k)$ for all k, then by Unambiguous Likelihood Sophistication they are indifferent according to the decision maker, that is $L \sim Q$. Therefore, the set of lotteries is equivalent to the set of vNM lotteries with probabilities of outcomes given by π-probabilities of events, and mixtures as defined in Definition 5.

Consecutive applications of Unambiguous Independence and Small Event Continuity imply that $\succsim$ on the set of lotteries satisfies Independence. Adding Order (S1) to Independence and Small Event Continuity yields that $\succsim$ on the set of lotteries is represented by a vNM utility function, denote it by u. We calibrate u so that $u(x_*) = 0$ and $u(x^*) = 1$. ■

Remark 1. *Applying Ambiguity Aversion consecutively implies that if $f, g \in \mathcal{F}$ are acts satisfying $f \succsim g$ then $\frac{k}{2^m} f \oplus (1 - \frac{k}{2^m}) g \succsim g$. Small Event Continuity then yields that the same is true for any $\alpha : 1 - \alpha$ mixture of f and g.*

Claim 2. *Let $f = [x_1, E_1; \ldots; x_m, E_m]$ be an act and L a lottery such that $f \sim L$. Then $\alpha f \oplus (1-\alpha) L \sim L$ for every $\alpha \in (0,1)$.*

Proof. Denote by $z_1, \ldots, z_t$ the outcomes involved in the lottery L, and by $\lambda_1, \ldots, \lambda_t$ their corresponding π-probabilities. Then,

$$L \sim [z_1, A_1^1; z_2, A_1^2; \ldots; z_t, A_1^t; \ldots; z_1, A_m^1; \ldots; z_t, A_m^t] \tag{1}$$

for events A_k^ℓ that satisfy $\cup_{\ell=1}^t A_k^\ell = E_k$ and $P_i(A_k^\ell) = \lambda_\ell P_i(E_k)$ for all i, for all ℓ and k. The claim is first proved for $\alpha = \frac{1}{2^n}$ for $n \in \mathbb{N}$, that is, for $\frac{1}{2^n} f \oplus (1 - \frac{1}{2^n}) L$. The proof is by induction on n.

For $n = 1$, Unambiguous Independence implies $\frac{1}{2}f \oplus \frac{1}{2}L \sim L$. Assume that $\frac{1}{2^n}f \oplus (1 - \frac{1}{2^n})L \sim L$. By Lyapunov's theorem, for every k and ℓ there are events $G_k^\ell \subseteq A_k^\ell$ with $P_i(G_k^\ell) = \frac{1}{2^n}P_i(A_k^\ell)$ for every i. The act which assigns x_k to G_k^ℓ for every ℓ, and z_ℓ to $A_k^\ell \setminus G_k^\ell$ for every k, is a $\frac{1}{2^n} : 1 - \frac{1}{2^n}$ mixture of f and L.

Again by Lyapunov's theorem, for each k and ℓ there are events $T_k^\ell \subseteq G_k^\ell$ that satisfy $P_i(T_k^\ell) = \frac{1}{2}P_i(G_k^\ell) = \frac{1}{2^{n+1}}P_i(A_k^\ell)$ for all i. The act which assigns x_k to T_k^ℓ for every ℓ and z_ℓ to $A_k^\ell \setminus T_k^\ell$ is, by its construction, a half-half mixture of $\frac{1}{2^n}f \oplus (1-\frac{1}{2^n})L$ and L, therefore by the induction assumption and Unambiguous Independence, it is indifferent to L. On the other hand, this same act is also a $\frac{1}{2^{n+1}} : 1 - \frac{1}{2^{n+1}}$ mixture of f and L, as it assigns x_k, the outcome under f, to $T_k^\ell \subseteq A_k^\ell$ for every ℓ, which satisfies $P_i(T_k^\ell) = \frac{1}{2^{n+1}}P_i(A_k^\ell)$, and z_ℓ, the outcome under L, to the complement of each A_k^ℓ. It is thus established that any $\frac{1}{2^{n+1}} : 1 - \frac{1}{2^{n+1}}$ mixture of f and L is indifferent to L.

Now suppose that g is a mixture $\frac{k}{2^n}f \oplus (1 - \frac{k}{2^n})L$. By Ambiguity Aversion, $g \succsim L$. We assume $g \succ L$ and derive a contradiction.

Let g' be a mixture in $\frac{1}{2}g \oplus \frac{1}{2}L$. Employing Unambiguous Independence, $g' \succ L$. However, there exists an act h such that h is a mixture of g and g' that at the same time is also a mixture $\frac{1}{2^{n'}}f \oplus (1-\frac{1}{2^{n'}})L$. By the previous paragraph $h \sim L$, whereas according to Ambiguity Aversion (see Remark 1) and the fact that $g, g' \succ L$ it should also hold that $h \succ L$. Contradiction. Therefore any $\frac{k}{2^n} : 1-\frac{k}{2^n}$ mixture of f and L is indifferent to L, and Small Event Continuity renders that the same is true for any $\alpha : 1-\alpha$ mixture. ■

Claim 3. *Let f and g be two acts, and L and Q two unambiguous acts. If $\frac{1}{2}f \oplus \frac{1}{2}Q \sim \frac{1}{2}g \oplus \frac{1}{2}L$, then $f \sim L$ if and only if $g \sim Q$.*

Proof. Suppose that $\frac{1}{2}f \oplus \frac{1}{2}Q \sim \frac{1}{2}g \oplus \frac{1}{2}L$ and $f \sim L$. By Unambiguous Independence, $f \sim L$ implies $\frac{1}{2}f \oplus \frac{1}{2}Q \sim \frac{1}{2}L \oplus \frac{1}{2}Q$. Suppose on the contrary that $g \succ Q$, then again by Unambiguous Independence, $\frac{1}{2}g \oplus \frac{1}{2}L \succ \frac{1}{2}Q \oplus \frac{1}{2}L$. Contradiction. ■

Claim 4. *Let f be an act and L, R lotteries such that $f \sim L$. Then for any $\alpha \in (0, 1)$, $\alpha f \oplus (1 - \alpha)R \sim \alpha L \oplus (1 - \alpha)R$.*

Proof. As in the previous proof, denote by $z_1, \ldots, z_t$ the outcomes involved in the lottery L, and by $\lambda_1, \ldots, \lambda_t$ their corresponding π-probabilities. Denote by $w_1, \ldots, w_n$ the outcomes obtained under R, with $\rho_1, \ldots, \rho_n$ their corresponding π-probabilities.

An $\alpha : 1-\alpha$ mixture of these lotteries is therefore a lottery which obtains outcomes $z_1, \ldots, z_t, w_1, \ldots, w_n$ with probabilities $\alpha\lambda_1, \ldots, \alpha\lambda_t, (1-\alpha)\rho_1, \ldots, (1-\alpha)\rho_n$.

As in the proof of Claim 2, let $f = [x_1, E_1; \ldots; x_m, E_m]$, and observe that L is indifferent to the act defined in (1) from that claim. Denote by g the act which assigns, for every k, the outcome x_k to events $G_k^\ell \subseteq A_k^\ell$, for all ℓ, where $P_i(G_k^\ell) = \alpha P_i(A_k^\ell)$ for all i, and the outcome z_ℓ, for each ℓ, to events $B_k^\ell = A_k^\ell \setminus G_k^\ell$, for every k. By its definition g is an $\alpha : 1-\alpha$ mixture of f and L, therefore by Claim 2, $g \sim L$.

Next consider an $\alpha : 1-\alpha$ mixture of L and R. For every k and ℓ, let the events $H_k^{\ell r}$, $r = 1, \ldots, t$ and $F_k^{\ell j}$, $j = 1, \ldots, n$, form together a partition of G_k^ℓ such that for every k, ℓ and r, $P_i(H_k^{\ell r}) = \alpha\lambda_r P_i(G_k^\ell)$, and for every k, ℓ and j, $P_i(F_k^{\ell j}) = (1-\alpha)\rho_j P_i(G_k^\ell)$, for all i. For every k and ℓ, let events $\hat{B}_k^{\ell r}$ for $r = 1, \ldots, t$ and events $\hat{C}_k^{\ell j}$, $j = 1, \ldots, n$ together form a partition of $A_k^\ell \setminus G_k^\ell$, such that for every k, ℓ and r, $P_i(\hat{B}_k^{\ell r}) = \alpha\lambda_r P_i(B_k^\ell)$, and for every k, ℓ and j, $P_i(\hat{C}_k^{\ell j}) = (1-\alpha)\rho_j P_i(B_k^\ell)$, for all i. The act which assigns z_r to $(\cup_{k,\ell} H_k^{\ell r}) \cup (\cup_{k,\ell} \hat{B}_k^{\ell r})$, and w_j to $(\cup_{k,\ell} F_k^{\ell j}) \cup (\cup_{k,\ell} \hat{C}_k^{\ell j})$ is an $\alpha : 1-\alpha$ mixture of L and R.

By cutting a half event (according to all the P_i's) out of every $H_k^{\ell r}$, and out of every $F_k^{\ell j}$, and assigning to it the outcome x_k, and similarly cutting a half-event out of every $\hat{B}_k^{\ell r}$, and a half-event from every $\hat{C}_k^{\ell j}$, and assigning to both the outcome z_ℓ, a half-mixture of g and $\alpha L \oplus (1-\alpha)R$ is obtained. This half-mixture assigns each outcome x_k to events that gain a total probability (according to all the P_i's) of $\frac{1}{2}\left(\sum_{\ell,r} P_i(H_k^{\ell r}) + \sum_{\ell,j} P_i(F_k^{\ell j})\right) = \frac{1}{2}\left(\sum_{\ell,r} \alpha\lambda_r P_i(G_k^\ell) + \sum_{\ell,j} (1-\alpha)\rho_j P_i(G_k^\ell)\right) = \frac{1}{2} P_i(G_k) = \frac{1}{2}\alpha P_i(E_k)$. It assigns each outcome w_j with probability $\frac{1}{2}(1-\alpha)\rho_j$, and each outcome z_ℓ with a total probability $\frac{1}{2}\alpha\lambda_\ell + \frac{1}{2}(\sum_{k,r} P_i(\hat{B}_k^{\ell r}) + \sum_{k,j} P_i(\hat{C}_k^{\ell j})) = \frac{1}{2}\alpha\lambda_\ell + \frac{1}{2}(\sum_{k,r} \alpha\lambda_r P_i(B_k^\ell) + sum_{k,j}(1-\alpha)\rho_j P_i(B_k^\ell)) = \frac{1}{2}\alpha\lambda_\ell + \frac{1}{2} P_i(B^\ell) = \frac{1}{2}\alpha\lambda_\ell + \frac{1}{2}(1-\alpha)\lambda_\ell = \frac{1}{2}\lambda_\ell$.

Similarly to the above, the lottery R is indifferent to an act

$$[w_1, C_1^1; w_2, C_1^2; \ldots; w_n, C_1^n; \ldots; w_1, C_m^1; \ldots; w_n, C_m^n] \quad ,$$

such that for every k the events C_k^j, $j = 1, \ldots, n$, partition E_k and satisfy $P_i(C_k^j) = \rho_j P_i(E_k)$ for every j and every i.

By Lyapunov's theorem, for every k and j there exist an event $T_k^j \subseteq C_k^j$ such that $P_i(T_k^j) = \alpha P_i(C_k^j)$. The act h which assigns, for every k, the outcome x_k to T_k^j for all

j, and for every j, the outcome w_j to $C_k^j \setminus T_k^j$ is, by its construction, an *alpha* : $1-\alpha$ mixture of f and R.

Consider a partition of T_k^j into sets $\hat{T}_k^j, D_k^{j\ell}, \ell = 1, \ldots, t$, such that for every k and j, $P_i(\hat{T}_k^j) = \frac{1}{2}P_i(T_k^j)$ and $P_i(D_k^{j\ell}) = \frac{1}{2}\lambda_\ell P_i(T_k^j)$. Let $\hat{C}_k^j, J_k^{j\ell}, \ell = 1, \ldots, t$ be a partition of $C_k^j \setminus T_k^j$ such that for every k and j, $P_i(\hat{C}_k^j) = \frac{1}{2}P_i(C_k^j \setminus T_k^j)$ and $P_i(J_k^{j\ell}) = \frac{1}{2}\lambda_\ell P_i(C_k^j \setminus T_k^j)$. The act which assigns, for every k, the outcome x_k to $\cup_j \hat{T}_k^j$, for every j, the outcome w_j to $\cup_k \hat{C}_k^j$, and for every ℓ assigns z_ℓ to the event $(\cup_{k,j} D_k^{j\ell}) \cup (\cup_{k,j} J_k^{j\ell})$, is a half-half mixture of $\alpha f \oplus (1-\alpha)R$ and L. It assigns each outcome x_k with probability, according to all i, $P_i(\cup_j \hat{T}_k^j) = \sum_j \frac{1}{2}P_i(T_k^j) = \sum_j \frac{1}{2}\alpha P_i(C_k^j) = \frac{1}{2}\alpha \sum_j \rho_j P_i(E_k) = \frac{1}{2}\alpha P_i(E_k)$, each outcome w_j with probability $P_i(\cup_k \hat{C}_k^j) = \sum_k \frac{1}{2}P_i(C_k^j \setminus T_k^j) = \sum_k \frac{1}{2}(1-\alpha)P_i(C_k^j) = \frac{1}{2}(1-\alpha)\sum_k \rho_j P_i(E_k) = \frac{1}{2}(1-\alpha)\rho_j$, and finally each outcome z_ℓ with total probability of $P_i\left((\cup_{k,j} D_k^{j\ell}) \cup (\cup_{k,j} J_k^{j\ell})\right) = \sum_{k,j} \frac{1}{2}\lambda_\ell P_i(T_k^j) + \sum_{k,j} \frac{1}{2}\lambda_\ell P_i(C_k^j \setminus T_k^j) = \sum_{k,j} \frac{1}{2}\lambda_\ell \alpha \rho_j P_i(E_k) + \sum_{k,j} \frac{1}{2}\lambda_\ell (1-\alpha)\rho_j P_i(E_k) = \frac{1}{2}\lambda_\ell$. This act therefore generates the same distribution over outcomes as does a half-mixture of g and $\alpha L \oplus (1-\alpha)R$, according to all the probabilities P_i. Unambiguous Likelihood Sophistication implies that the two acts are indifferent. As $g \sim L$, then Claim 3 implies that $\alpha f \oplus (1-\alpha)R \sim \alpha L \oplus (1-\alpha)R$.

For any two acts f and g, Small event continuity and Monotonicity imply that there are lotteries L and Q such that $f \sim L$ and $g \sim Q$. Therefore for any lottery R and $0 < \alpha \leq 1$, the above and the independence of $\succsim$ over lotteries yield,

$$\begin{aligned} f \succsim g \iff L \succsim Q \iff \\ \alpha L \oplus (1-\alpha)R \succsim \alpha Q \oplus (1-\alpha)R \iff \\ \alpha f \oplus (1-\alpha)R \succsim \alpha g \oplus (1-\alpha)R \end{aligned}$$

■

The relation $\succsim$ on the set of acts $\mathcal{F}$ may be extended to the set of Anscombe-Aumann (1963) acts over (S, Σ) (as rephrased by Fishburn, 1970; shortened to *AA acts*) in the following manner, where only simple acts (which return finitely many lotteries) are considered.

Suppose an AA act a over the state space S, returning lottery $L = (x_1, p_1; \ldots; x_t, p_t)$

on event E. The lottery L can be implemented on E by assigning the outcomes $x_1, \ldots, x_t$ to a partition $\{E_1, \ldots, E_t\}$ of E respectively, where for all j, $P_i(E_j) = p_j P_i(E)$ for all i. The existence of such events E_j is guaranteed by Lyapunov's theorem, and every choice of appropriate events generates acts which are indifferent following Unambiguous Likelihood Sophistication. An act in $\mathcal{F}$ returning for every event E the implementation of the lottery that a returns on E is said to be a Savage act matching a.

For every pair of AA-acts a and b, the ranking of these acts according to the extended relation $\succsim$ is defined by $a \succsim b$ if and only if $f \succsim g$, where f is a Savage act matching a and g is a Savage act matching b. Observe yet again that all the Savage acts matching the same AA-act are indifferent on account of Unambiguous Likelihood Sophistication. The extended relation $\succsim$ on AA-acts is therefore well defined.

Specifically, any vNM lottery (a constant act in the AA framework), $(x_1, p_1; \ldots; x_m, p_m)$, is matched with a lottery act (an $\mathcal{E}$-measurable act), $[x_1, E_1; \ldots; x_m, E_m]$, with $\pi(E_k) = p_k$. For two vNM lotteries a and b, $a \succsim b$ if and only if $u(L) \geq u(Q)$, where L and Q are lottery acts matching a and b, respectively. The extended $\succsim$ on vNM lotteries is thus represented by the vNM functional u (see Claim 1).

The extended relation $\succsim$ over AA-acts satisfies all the axioms of the Maxmin Expected Utility model of Gilboa and Schmeidler (1989, henceforth GS), with mixtures as defined in Definition 5. First, Weak Order (S1) is explicitly assumed and Non-Degeneracy follows from Minimal Agreement. The Continuity assumption used in GS follows from Small Event Continuity (S3). The next claim proves that Monotonicity as in GS holds. As this paper deals only with acts which return finitely many values it suffices to show that Monotonicity is satisfied whenever the concerned AA-acts differ by one returned lottery.

Claim 5. *Let $a = [L_1, A_1; L_2, A_2; \ldots; L_t, A_t]$ and $b = [Q_1, A_1; L_2, A_2; \ldots; L_t, A_t]$ be two AA-acts. Then $L_1 \succsim Q_1$ implies $a \succsim b$.*

Proof. Suppose that $L_1 \succsim Q_1$. By definition of the extended relation $\succsim$ the same ranking holds true for the lottery acts that implement L_1 and Q_1. If L_1 and Q_1 involve only rational-valued probabilities then those matching lottery acts are unambiguous by the definition, and the Savage acts matching a and b are unambiguous on A_1. The ranking of the lottery acts therefore implies $a \succsim b$ through Extended Monotonicity (S2) and the definition of the extended $\succsim$ on AA-acts.

Otherwise L_1 and Q_1 involve irrational probabilities. Let f_1 and g_1 denote the (Savage) lottery acts that match the vNM lotteries L_1 and Q_1. If $L_1 \succ Q_1$ then

$f_1 \succ g_1$, and by Small Event Continuity and standard monotonicity (which is implied by Extended Monotonicity, S2) there exist arbitrarily close rational-valued lottery acts that satisfy this preference. The preference for the acts, $[L_1, A_1; \ldots; L_t, A_t] \succsim [Q_1, A_1; L_2, A_2; \ldots; L_t, A_t]$, follows using S2 and the definition of the extended $\succsim$ (again standard monotonicity is employed).

Finally suppose that $L_1 \sim Q_1$, where L_1 and Q_1 involve irrational probabilities. If still $[L_1, A_1; \ldots; L_t, A_t] \succ [Q_1, A_1; L_2, A_2; \ldots; L_t, A_t]$ then by Small Event Continuity lotteries L_1' and Q_1' with rational-valued probabilities can be found, such that $L_1 \succsim L_1'$ and $Q_1' \succsim Q_1$ by standard monotonicity, yet a preference $[L_1', A_1; \ldots; L_t, A_t] \succ [Q_1', A_1; L_2, A_2; \ldots; L_t, A_t]$ still holds. By S2 applied on matching Savage acts this implies $L_1' \succ Q_1'$, thus by transitivity $L_1 \succ Q_1$. Contradiction. ■

Lastly, Uncertainty Aversion and Certainty Independence follow from Remark 1 and Claim 4. Since $\succsim$ as defined on the set of AA-acts satisfies all the axioms of GS, a Maxmin Expected Utility representation on these acts follows. That is, there exists a unique closed and convex set of probabilities $\mathcal{C}$ over Σ such that for every two AA acts, a and b,

$$a \succsim b \quad \Leftrightarrow \quad \min_{p \in \mathcal{C}} \mathbb{E}_p(u \cdot a) \geq \min_{p \in \mathcal{C}} \mathbb{E}_p(u \cdot b) \ . \tag{2}$$

In particular, for every two acts $f, g \in \mathcal{F}$,

$$f \succsim g \quad \Leftrightarrow \quad \min_{p \in \mathcal{C}} \mathbb{E}_p(u \cdot f) \geq \min_{p \in \mathcal{C}} \mathbb{E}_p(u \cdot g) \ .$$

Monotone Continuity is next invoked to obtain that all the probabilities in $\mathcal{C}$ are countably additive.

Claim 6. *All the probabilities in $\mathcal{C}$ are countably additive.*

Proof. Let A_n be a sequence of events increasing to S. For $m \in \mathbb{N}$, take a dyadic unambiguous partition $\{E_1, \ldots, E_{2^m}\}$. Consider the half-events of S measurable w.r.t. this partition, and then their half-events measurable w.r.t. this partition, and so on. Every individual preference finds every such pair of half-events to be equally likely, therefore by No Added Ambiguity the same holds for the decision maker's preference. Thus for every k, $p(E_k) = 1/2^m$ for all $p \in \mathcal{C}$. By Monotone Continuity, $\min_{p \in \mathcal{C}} p(A_n) > 1 - 1/2^m$, thus $p(A_n) > 1 - 1/2^m$ for all $p \in \mathcal{C}$, from some index n_0 and on. It follows that $p(A_n) \longrightarrow 1$ for every $p \in \mathcal{C}$, hence all the probabilities in $\mathcal{C}$ are countably additive.

■

Note that by the representation on lotteries by a vNM functional, for any event $E \in \mathcal{E}$ such that $\pi(E) = 0.5$, $\succsim$ evaluates x^*Ex_* by $\min_{p\in\mathcal{C}} \mathbb{E}_p(u(x^*Ex_*)) = 0.5u_0(x^*) + 0.5u_0(x_*)$. The fact that $\mathcal{C}$ is contained in the convex hull of $\{P_1, \ldots, P_n\}$ follow from Theorem 1 in Alon and Gayer (2015), by employing No Added Ambiguity which takes the place of Likelihood Pareto in that related paper.

4.2 The axioms are necessary: (ii) implies (i)

Weak Order (S1) follows easily from the representation. No Added Ambiguity is again shown to be implied in Theorem 1 in Alon and Gayer (2015). For Unambiguous Likelihood Sophistication, suppose two acts, $f = [x_1, F_1; \ldots; x_m, F_m]$ and $g = [x_1, G_1; \ldots; x_m, G_m]$, where $P_i(F_k) = P_i(G_k)$ for every i and k. Consider a probability $p \in \mathcal{C}$. Part (ii) of the theorem states that $p = \sum_{i=1}^n \lambda_i P_i$, for nonnegative coefficients λ_i such that $\sum_{i=1}^n \lambda_i = 1$, hence,

$$p(F_k) = \sum_{i=1}^n \lambda_i P_i(F_k) = \sum_{i=1}^n \lambda_i P_i(G_k) = p(G_k) \ .$$

That is, the two acts are translated to the same distribution over outcomes according to every prior in the set $\mathcal{C}$, thus they obtain the same minimal expected utility on that set, yielding that $f \sim g$.

To show that Extended Monotonicity (S2) is satisfied consider an event E and two unambiguous acts, $L = [x_1, E_1; \ldots; x_m, E_m]$ and $Q = [y_1, E_1; \ldots; y_m, E_m]$, such that $\{E_1 \cap E, \ldots, E_m \cap E\}$ is an unambiguous partition of E. On the one hand, the preference $L \succsim Q$ holds if and only if $\frac{1}{m}\sum_{k=1}^m u(x_k) \geq \frac{1}{m}\sum_{k=1}^m u(y_k)$. On the other hand, according to all the probabilities P_i, thus by every probability $p \in \mathcal{C}$, $p(E_k \cap E) = \frac{1}{m}p(E)$. Therefore for every $p \in \mathcal{C}$ and any act h, the expected utilities of LEh and QEh according to p are:

$$\begin{aligned}
\mathbb{E}_p(u \cdot (LEh)) &= p(E)\frac{1}{m}\sum_{k=1}^m u(x_k) + (1 - p(E))\mathbb{E}_p(u \cdot h|E^c) \ , \text{ and} \\
\mathbb{E}_p(u \cdot (QEh)) &= p(E)\frac{1}{m}\sum_{k=1}^m u(y_k) + (1 - p(E))\mathbb{E}_p(u \cdot h|E^c) \ .
\end{aligned}$$

Hence $L \succsim Q$ implies $\mathbb{E}_p(u \cdot (LEh)) \geq \mathbb{E}_p(u \cdot (QEh))$ for all $p \in \mathcal{C}$, and the inequality for the minimum follows, yielding $LEh \succsim QEh$.

For Small Event Continuity (S3) note that by Lyapunov's theorem for every $f = [x_1, F_1; \ldots; x_k, F_k]$ and every m there exists an m-partition of each of the events F_ℓ, $\{F_\ell^j\}_{j=1}^m$, to m equally likely events according to all the probabilities P_i, thus according to all the probabilities in $\mathcal{C}$. Applying such a partition for a large enough m yields a partition $\{E_1, \ldots, Em\}$ of S with $E_j = \bigcup_{\ell=1}^{k} F_\ell^j$ that yields the required result.

Ambiguity Aversion (S5) and Unambiguous Independence (S6) follow from the minimum functional, and the fact that all the prior probabilities in $\mathcal{C}$ agree on the probability of events in an unambiguous partition and on the distribution over outcomes generated by unambiguous acts.

For Monotone Continuity (S4) let A_n be a sequence of events increasing to S, and let E be a non-null event. Non-nullity of E renders $\max_{p\in\mathcal{C}} p(E) > 0$, hence the bet $x^*(S \setminus E)x_*$ obtains a value of $1 - \max_{p\in\mathcal{C}} p(E)$ which is smaller than 1. According to Theorem 1.xi in Maccheroni and Marinacci (2001), since $\mathcal{C}$ is closed and bounded the limit $\lim_{n\to\infty} p(A_n) = 1$ is uniform in it, that is, for $1 - \max_{p\in\mathcal{C}} p(E)$ there exists an n_0 such that for all $n > n_0$, $p(A_n) > 1 - \max_{p\in\mathcal{C}} p(E)$ for all $p \in \mathcal{C}$. Since the minimum is attained on $\mathcal{C}$ it follows that $\min_{p\in\mathcal{C}} p(A_n) > 1 - \max_{p\in\mathcal{C}} p(E)$ for all $n > n_0$, and Monotone Continuity is established. ■

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