Inductive Inference with Incompleteness*

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Abstract

We present an axiomatic model of a process wherein likelihoods of eventualities are compared based on data. One eventuality is perceived as more likely than another whenever the data corroborates this conclusion. However, the correct relevance of records to the eventualities under consideration may be impossible to ascertain with any degree of surety due to multiple interpretations of the data, formalized by allowing the evaluator to entertain multiple weighting functions. The evaluator ranks one eventuality as more likely than another whenever its total weight over the entire database is higher, according to all relevance-weighting functions. Otherwise, the comparison is indecisive.

Keywords: Incompleteness, Inductive Inference, Case-Based Decision Theory, Likelihood Comparisons.

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1 Introduction

1.1 Motivation

There are many instances where people are faced with problems in which they have to compare the likelihood of several eventualities. These problems can range from forecasting the weather to predicting the chances of survival of the Eurozone or the chances of reaching a peace agreement in the Middle East. What is an advisable process of performing such likelihood comparisons? Is it reasonable to require that the process always determine whether one eventuality is more likely than another?

The present work, focusing on evaluation methods based on past observations, rests upon the widely accepted principle that similar circumstances lead to similar outcomes. One eventuality is considered to be more likely than another if the data corroborate the occurrence of the former more than they do the latter. An example of this type of quandary is that of a doctor trying to diagnose a patient's condition. It would be simple for the doctor to reach a diagnosis in case he or she has seen many patients with the exact symptoms as those of the current patient that were successfully diagnosed. Now suppose that the current patient suffers from many nonspecific symptoms such as pain, fatigue, and dizziness, symptoms that are common to many medical conditions. The doctor has seen patients who exhibited part of these symptoms, but not a single patient who suffered from all of them. Thus the current patient may be suffering from any one of the illnesses of these past patients or even an illness unfamiliar to the doctor. In such a situation the doctor may feel uncertain as to the relevance of these past cases to the current case, and consequently will refrain from making a diagnosis.

This description of a patient with unexplained physical symptoms is not uncommon to the medical field (with estimates ranging from 15% to 30% of the primary care consultations; see Kirmayer et al., 2004). Kirmayer et al. (2004) mention several reasons for the difficulty to diagnose patients, among them non-measurability of symptoms (e.g. fatigue) that could be attributed to several medical conditions. Support groups of undiagnosed patients try to promote the formation of a database gathered from people who were diagnosed in order to improve the diagnosis process.[1,2]

It appears that similar difficulties in making comparisons are present in other fields of medicine. The US Preventive Services Task Force (USPSTF) issues guidelines for preventive health care services which are broken down by age, gender, medical condition and other criteria.[3] These guidelines are based on evidence that is gathered from multiple research studies. Some of these studies were conducted on special populations that differ from the population for which guidelines are intended. Therefore, the task force determines conditions under which extrapolation and generalization are reasonable, explicitly including similarities of the populations studied and the population in question. However for some issues the USPSTF publishes 'I statements' where no recommendation is made, when it concludes that studies are of poor quality or when they produce conflicting results that do not permit conclusions about likely benefits and harms (Harris et al., 2001). In such cases the USPSTF advises that patients be informed about the uncertain balance of benefits and harms, stating that its recommendation may be altered after gathering more specific observations. Quoting Harris et al. (2001, p. 27), "Even well-designed and well-conducted studies may not supply the evidence needed if the studies examine a highly selected population of little relevance to the general population seen in primary care". The use of 'I statements' therefore appears to imply that uncertainty regarding the relevance of existing data is one of the primary causes for not making recommendations.

The evaluation process described above contains two major components: first that the comparison of likelihoods of eventualities depends on past observations, and second that the evaluator may refrain from comparisons due to the existence of several plausible methods to analyze or interpret the data. These different interpretations may lead to diametrically opposite conclusions which would render impossible any comparison of the likelihoods of the eventualities under consideration. Doctors abstain from making recommendations to undergo screening tests when they cannot assess the

[1] See http://www.inod.org/missionobjectives0.html for such a support group's list of objectives.

[2] The National Institute of Health (NIH) in the US recognized the severeness of this problem and initiated the Undiagnosed Diseases Program in 2008 to help people who have long eluded diagnosis.

[3] See the USPSTF website, http://www.uspreventiveservicestaskforce.org. We thank Irit Dror, MD, for the USPSTF example.

benefits and risks, and similarly refrain from making a diagnosis when it is questionable. Surely in the above medical example it would be preferable that the patient be told that the doctor could not assess which illness is more likely, rather than being misled by an arbitrary, poorly-based diagnosis.[4]

On a related note, there are many situations in which there is an individual or organization that makes a decision based on evaluations of likelihoods of a separate party. In those situations the decision maker will generally entertain various considerations and constraints, of which comparison of likelihoods is but one element. If the evaluator were to find it impossible to compare some likelihoods, then such difficulty is in itself informative. The decision maker would typically prefer that this predicament be revealed rather than have it papered over by an unfounded conclusion. Put differently, not requiring that every likelihood comparison be determined yields a more flexible decision process, leaving room for additional normative criteria (for example, the decision maker may conclude that more data needs to be collected before making a decision).

This is one of the reasons for the USPSTF published 'I statements'. Quoting Harris et al. (2001, p. 31):

> ...the Task Force has taken a conservative, evidence-based approach ... refraining from making recommendations when they cannot be supported by evidence. This is done with the understanding that clinicians and policymakers must still consider additional factors in making their own decisions.

The purpose of this essay is to describe an evaluation process wherein data-based comparisons of likelihoods are performed, but which also allows for non-determinability. The data are comprised of separate past cases, whose relevance to the case that is being examined varies, and may be unclear. Eventualities are compared by evaluating the

[4]If the doctor could assign a weight to each interpretations of the data (standing for the probability that it obtains), his or her ranking of eventualities could be completed using these second-order weights. Incompleteness emerges precisely when the evaluator cannot form such weights (as is the case in Bewley (2002)).

weight lent by the database to their likelihoods of occurrence in the current case. While there may be varied reasons for failing to make a comparison, in the problems we have in mind incomparability is caused by conflicting interpretations of the data. When several plausible interpretations of the data lead to opposite conclusions, inability to make a valid comparison obtains. On the other hand, if all interpretations of the data are aligned a ranking of likelihoods is issued.

1.2 The Model

The model of Inductive Inference (Gilboa and Schmeidler, 2003; henceforth abbreviated GS) serves as alternative to expected utility paradigm, that can address problems of choice under uncertainty, wherein, both the states of nature and the possible outcomes are hard to comprehend.[5] Instead of forming beliefs over the states of nature, as suggested by the expected utility paradigm, decision makers draw upon their experience in similar past cases to help them reach decisions. According to GS a weight is assigned to each record in the database expressing its relevance to the eventuality under consideration. One eventuality is ranked above another if its aggregate weight across the database is higher. For instance, when trying to diagnose a patient, a past case of a patient with similar symptoms to those of the current case, who suffered from illness A, is more relevant for the eventuality 'illness A' than it is for the eventuality 'illness B'. Accordingly, in the representation this record will attribute more weight to 'illness A' than to 'illness B'. Typically this weight will be higher the more resemblance there is between the symptoms of the past case and those of the current patient. Illness A will be considered more likely than B whenever its total weight across the database is higher.

Ranking eventualities according to the GS model requires the capability of assigning an *exact* weight, expressing degree of relevance, to each pair of eventuality and record in the database. This is a reasonable assumption when a good understanding exists of the factors responsible for relevance of past cases to the current situation. However,

[5]The Inductive Inference model is closely related to Gilboa and Schmeidler's Case-Based Decision theory, 1995 and 1997.

when the situation at hand is not completely understood or perhaps when there is a lot at stake, the evaluator may be reluctant to commit to specific weights. For instance, the doctor in the medical example above may not be able to determine whether a patient suffering from pain and fatigue resembles the current patient more than does a patient suffering from fatigue and dizziness, if the current patient exhibits all three symptoms.

The present work generalizes the GS model by allowing for a *set* of relevance-weights, thus accommodating an evaluation process of an individual who is unsure about the relevance of records in the database to eventualities under consideration. In the model one eventuality is considered more likely than another, if and only if, this ranking is corroborated by the data according to every relevance-weighting function. If an eventuality is evaluated as more likely than another according to one relevance-weighting, but according to another relevance-weighting the opposite is true, then the evaluator will refrain from issuing a comparison of the likelihoods of the two eventualities. The model thus converts lack of confidence into a sign of caution. Multiplicity of weighting functions is a reflection of there being different interpretations of the data, and incomparability arises when conflicting interpretations of the data obtain.

Formally, a relevance-weighting function v assigns a weight to each pair of an eventuality and a record in a database. For two eventualities, x and y, and a database I, x is considered more likely than y given database I, if and only if, for *every* relevance-weighting function v in some set V,

$$\sum_{c \in I} v(x, c) \geq \sum_{c \in I} v(y, c) \tag{1}$$

where c denotes a record in the database, and $v(x, c)$ is the weight which record c provides to eventuality x, according to the function v. In other words, x is more likely than y given database I, if and only if, there is unanimity over V that I lends more weight to eventuality x than to eventuality y. Therefore it may happen that, given database I, neither x is more likely than y nor y is more likely than x.

The main result of the paper is an axiomatic characterization of the representation in (1).

In comparison to the GS model completeness is suppressed. Also, diversity which is a richness condition is disposed of. Diversity is less suitable to describe incompleteness as it requires many comparisons to be completed. On the other hand, GS's Combination and Archimedean axioms are replaced by a stronger axiom which is termed Independence of Relevance. In a complementary result, our axioms are supplemented by completeness in order to obtain a representation as in (1) with a single relevance-weighting function v.

1.3 Related Literature

There is a vast body of literature within decision theory providing axiomatizations of relations that, due to ambiguity, are allowed to be incomplete. Ambiguity in these models is represented by a decision maker having a set of priors over the states of nature, which is analogous to our evaluator considering a set of relevance-weights of past records to eventualities. The decision maker prefers one alternative over another, if and only if, this alternative is preferred according to each prior in the set of priors (see, for instance, Giron and Rios (1980), Bewley (2002), Gilboa et al. (2010), and Galaabaatar and Karni (2011)). However, in these works the preferences do not explicitly depend on data. By contrast, the ability of the evaluator in the present model to rank the likelihoods of two eventualities is a function of the available data, which is a primitive of the model.

Ambiguity was introduced into the Inductive Inference (or Case-Based) framework by Eichberger and Guerdjikova (2010 and 2013), first in the context of belief formation and then in the context of decision-making. The decision-making model suggests an α-maxmin rule which is complete. Eichberger and Guerdjikova (2010 and 2013) refer to two types of ambiguity: the first, due to the small number of observations in the database, where absence of data gives rise to the largest amount of ambiguity that gradually disappears as the number of observations grows. This aspect is not part of the present model, for incomparability is only the result of contradictory interpretations, which may or may not obtain, regardless of the size of the database. The second source of ambiguity mentioned in Eichberger and Guerdjikova (2010 and 2013) is caused by uncertainty regarding the relevance of past observations to the current problem. This second type is in line with the reasoning considered here and does not disappear with the mere accumulation of data.

1.4 Outline of the Paper

The setup and assumptions are described in Section 2. Section 3 contains the main theorem of the paper, a unanimity representation of incomplete relations, followed by a representation when relations are assumed to be complete. Section 4 concludes, and Section 5 contains the proofs.

2 Setup and Assumptions

2.1 Setup and Notation

$\mathbb{X}$	a finite, non-empty set of eventualities, with typical elements $x, y, \ldots$.
$\mathbb{C}$	a finite, non-empty set of record types, with a typical element c. This is the set of all possible classes of past observations that can be found in a database.
$\mathbb{J} = \mathbb{Z}_+^{\mathbb{C}}$	the set of all databases, which are functions from record types to nonnegative integers, with typical elements $I, J, \ldots$. For $I \in \mathbb{J}$ and $c \in \mathbb{C}$, $I(c)$ denotes the number of times record type c appears in database I.
$\{\succsim_I\}_{I \in \mathbb{J}}$	for a database I, $\succsim_I$ is a binary relation over $\mathbb{X}$. A ranking $x \succsim_I y$ means that eventuality x is at least as likely as eventuality y, given database I.

In the medical example above the eventualities may be ‘illness A’ and ‘illness B’, and a record type consists of various symptoms and the illness of the patient, so for example the record type (‘no pain’, ‘yes fatigue’, ‘illness A’) represents a case of a patient with no pain, who complained about fatigue, and suffered from illness A. A database is a summary of the number of times the doctor has encountered each record type. Given a database the doctor tries to evaluate whether illness A is more likely than illness B. The ranking 'illness A' $\succsim_I$ 'illness B' implies that the doctor maintains that illness A is at least as likely as illness B given database I. Lack of both 'illness A' $\succsim_I$ 'illness B' and 'illness B' $\succsim_I$ 'illness A' indicates that the doctor is unable to assess which of the two illnesses is more likely. For two databases I and J, $I + J$ is the database obtained by pointwise addition, namely, $(I + J)(c) = I(c) + J(c)$ for every $c \in \mathbb{C}$.

2.2 Assumptions

First and foremost, the model does not impose completeness of relations $\{\succsim_I\}_{I\in\mathbb{J}}$. Without completeness, weak relations allow to distinguish between pairs of eventualities that are equally likely and those whose likelihoods are incomparable. We, therefore, consider weak relations, for which reflexivity should hold.

A1. Reflexivity: For every eventuality x and database $I \in \mathbb{J}$, $x \succsim_I x$.

Next we discuss a few standard assumptions, demonstrate that they are insufficient, and introduce the set of necessary and sufficient axioms employed in our model.

2.2.1 *Combination and Related Axioms*

A standard assumption presented first is transitivity, which states that for every three eventualities x, y, z and any database I, if both $x \succsim_I y$ and $y \succsim_I z$ then $x \succsim_I z$. Transitivity is a basic condition expressing that the extent to which a database corroborates an eventuality is independent of the eventuality to which it is being compared. It would fail, for instance, if different criteria were used to compare different eventualities.

To gain intuition for the kind of independence of relevance across eventualities that is entailed by transitivity, consider an evaluator who wishes to predict who will win the NYC marathon, given contestants' previous race times. The evaluator uses the following method to rank contestants: two contestants who have already competed in the NYC marathon are ranked according to their average finishing times in the NYC marathon, otherwise they are ranked according to their average finishing times in all marathons. Now suppose that only contestants A and B have participated in a NYC marathon and that A's average finishing time is better than B's in those marathons. However, when considering all marathon finishing times, B's average is better than contestant C's, and C's average is better than A's.

In this case, the evaluator would conclude that A is more likely to win this year's NYC marathon than B, and B is more likely to win than C, who is more likely to win than A, a conclusion that violates transitivity. Transitivity fails in this example, because in contrast to the notion of independence of relevance across eventualities, the relevance of past observations to an eventuality depends on the eventuality to which it is being compared. When comparing A and B, the only relevant observations are those from prior NYC marathons, while the other observations become relevant only when comparing either of them to contestant C who has not participated in a NYC marathon.

Another assumption, central to the GS model, is Combination, which states that if eventuality x is evaluated as more likely than eventuality y given two separate databases I and J, then x should also be evaluated as more likely than y when the two databases are combined. Formally,

Combination: For any two eventualities x and y and databases I and J, if $x \succsim_I y$ and $x \succsim_J y$ then $x \succsim_{I+J} y$. If, in addition, either $\neg(y \succsim_I x)$ or $\neg(y \succsim_J x)$, then $\neg(y \succsim_{I+J} x)$.

(If completeness is assumed, the negations in the second part convert into strict rankings).[6]

Combination requires that the relevance of any one record in a database to an

[6] A similar axiom appears in Young (1975) in the context of social choice, that employs repetitions of preference orders over a finite set of alternatives, so that each preference order corresponds to a record type in our setup.

eventuality be independent of all the other records in that database. So, analogously to the above, it entails *independence of relevance across records*. For a better understanding of this axiom consider an example, taken from GS, where it fails. Suppose an individual trying to compare the likelihoods of a coin being fair or biased using a database of past tosses. The corroboration that a record of tails provides to the likelihood that the coin is biased is higher when the other records in the database are mainly tails than it is when the database contains a nearly equal number of records of heads and tails. The corroboration that a 'tails' record lends to the likelihood of 'biased coin' therefore depends upon the composition of other records in the database (see GS for a detailed discussion of this point and more examples).

Under the assumption of completeness, Combination implies several other properties that do not follow when completeness is suppressed. As will be made clear, these properties match the type of ambiguity we seek to address in our model and are, like Combination, consistent with the idea of relevance being independent across records.

For the special case of $I = J$, Combination implies that if $x \succsim_I y$ then $x \succsim_{2I} y$ and together with completeness the opposite is also implied, namely if $x \succsim_{2I} y$, then $x \succsim_I y$, or, equivalently, if $\neg(x \succsim_I y)$ then $\neg(x \succsim_{2I} y)$. Without completeness the second implication does not necessarily follow. It seems reasonable to impose this property in our model, as it matches the underlying assumption that inability to make comparisons is due to conflicting interpretations of the data. In particular, when $\neg(x \succsim_I y)$ there must be one interpretation of the records in I whereby y is evaluated as more likely than x. Independence of relevance across records means that in the duplicated database, $2I$, each copy of I maintains its original relevance, hence the same interpretations that apply for I apply for $2I$ as well, yielding $\neg(x \succsim_{2I} y)$.

The next axiom, termed Replication-Invariance of Incomparability, expresses the above idea (generalized to any number of replications of the data).

A2. Replication-Invariance of Incomparability: For any two eventualities x and y, database $I \in \mathbb{J}$ and an integer $n \in \mathbb{N}$, if $\neg(x \succsim_I y)$ then $\neg(x \succsim_{nI} y)$.

According to this axiom incompleteness cannot be ascribed to there being an insufficient number of records per se. The model is therefore inappropriate for describing the problem of a statistician that infers that one alternative is more likely than another when that hypothesis is statistically significant. Typically for such inference, for a very small database neither can the claim x is more likely than y be refuted nor can the opposite. Yet, one of these assertions will become significant for a large enough number of replications of the database, thus violating Replication-Invariance of Incomparability.

In the presence of completeness, Combination also entails that if $x \succsim_{I+J} y$ and $y \succsim_I x$ then $x \succsim_J y$. However, when completeness is not imposed, inability to compare x with y on J is still consistent with Combination. Nevertheless, the requirement that $x \succsim_J y$ once again conforms to independence of relevance across records, as the corroboration that records in I lend to the likelihood of y compared with that of x cannot be altered by records in J. Since y is perceived to be more likely than x on I, if this evaluation is reversed on $I + J$, then it must be the records in J that caused this reversal by corroborating x being more likely than y. The following axiom incorporates this idea.

Strong Combination: For any two eventualities x and y and databases I and J,

If $x \succsim_I y$ and $x \succsim_J y$, then $x \succsim_{I+J} y$.

If $x \succsim_{I+J} y$ and $y \succsim_I x$, then $x \succsim_J y$.

2.2.2 *Insufficiency of the Above Axioms*

Reflexivity, transitivity, Replication-Invariance of Incomparability, and Strong Combination are obviously necessary conditions for the representation (1). However, they are insufficient, as demonstrated by the following example.

Example 1. Let $\mathbb{X} = \{x, y, z, w\}$ and $\mathbb{C} = \{1, 2\}$. Suppose that the relations $\succsim_I$ are all reflexive, and that

$$x \succsim_{(1,0)} y \quad , \quad x \succsim_{(1,1)} z \quad , \quad x \succsim_{(0,1)} w \quad ,$$
$$z \succsim_{(1,0)} w \quad , \quad w \succsim_{(1,1)} y \quad , \quad z \succsim_{(0,1)} y \quad , \quad \text{and}$$
$$\neg(x \succsim_{(2,2)} y) \quad .$$

It is easy to see that transitivity, Replication-Invariance of Incomparability, and Strong Combination are all satisfied,[7] yet these rankings cannot be represented by (1). Assuming there existed such a representation, the rankings above would imply that for all $v \in V$,

$$\begin{aligned} v(x,1) &\geq v(y,1) \ , \\ v(z,1) &\geq v(w,1) \ , \\ v(x,1) + v(x,2) &\geq v(z,1) + v(z,2) \ , \\ v(w,1) + v(w,2) &\geq v(y,1) + v(y,2) \ , \\ v(x,2) &\geq v(w,2) \ , \text{ and} \\ v(z,2) &\geq v(y,2) \ . \end{aligned}$$

By summing over these inequalities $2v(x,1)+2v(x,2) \geq 2v(y,1)+2v(y,2)$ for all $v \in V$. Hence by the representation $x \succsim_{(2,2)} y$, contradicting $\neg(x \succsim_{(2,2)} y)$.

Example 1 not only demonstrates that the axioms discussed above are too weak to derive the representation in (1), but also raises the question of whether it is reasonable to require that a stronger condition hold. In particular we need to examine whether, given the impetus of the model, the rankings stated in the example should in fact imply that $x \succsim_{(2,2)} y$. We argue the affirmative. On database $(1,1)$ it is possible to complete the ranking between z and w in two ways, either by $z \succsim_{(1,1)} w$ or by $w \succsim_{(1,1)} z$. In case $z \succsim_{(1,1)} w$, transitivity implies $x \succsim_{(1,1)} y$, and by Combination it follows that $x \succsim_{(2,2)} y$. Otherwise, if $w \succsim_{(1,1)} z$, then by Strong Combination $w \succsim_{(0,1)} z$ leading to $x \succsim_{(0,1)} y$ by transitivity. Then by applying Combination twice, both $x \succsim_{(1,1)} y$ and $x \succsim_{(2,2)} y$. The

[7] For this purpose assume that rankings on replications of the databases mentioned above are properly completed according to Combination. Moreover, rankings can be added in order to satisfy practically any form of continuity (such as GS Continuity, defined in Section 3.1).

ranking of eventualities z and w on $(1,1)$ may not be known, yet if all possible rankings lead to the same conclusion that $x \succsim_{(2,2)} y$ then this ranking should be completed in that manner.

2.2.3 Exact Independence of Relevance

The previous discussion suggests that the strengthening of the axioms introduced above is in order. We, therefore, introduce a new condition, which states that the relevance of a record to an eventuality is independent of the other records and the other eventualities, in a stronger sense than as entailed by the conditions discussed so far. The impetus for independence across records is the same as that mentioned above in the discussion of Combination, where the relevance of a record to an eventuality is not altered by other records in the database. Independence across eventualities means that the relevance of a record to an eventuality remains the same, regardless of the eventuality to which the comparison is made, a notion expressed by transitivity (see the example of the NYC marathon above).

Notice that in Example 1, x is never less likely than another eventuality and y is never more likely than another eventuality, and the rankings satisfy the following conditions: the aggregation of all records on which x is more likely than another eventuality (database $K = (2,2)$, which is the aggregation of $(1,0)$, $(1,1)$, and $(0,1)$) equals the aggregation of all records on which y is less likely than another eventuality. Moreover, for any eventuality other than x and y, the aggregation of all records on which it is more likely is the same as the aggregation of records on which it is less likely. The next axiom allows us to deduce, then, that $x \succsim_{(2,2)} y$.

For a general formulation, let $\mathcal{R} = \{m_1 \succsim_{J_1} \ell_1, \ldots, m_n \succsim_{J_n} \ell_n\}$ denote some set of rankings (which need not include every ranking that holds, or even every eventuality in the model), and define, for eventuality z, $\mathcal{M}_{\mathcal{R}}(z) = \sum_{t:m_t=z} J_t$, and, $\mathcal{L}_{\mathcal{R}}(z) = \sum_{t:\ell_t=z} J_t$. That is, $\mathcal{M}_{\mathcal{R}}(z)$ aggregates all records in $\mathcal{R}$ for which z is considered more likely (compared to some eventuality), and likewise $\mathcal{L}_{\mathcal{R}}(z)$ aggregates all those records in $\mathcal{R}$ for which it is considered less likely. Both $\mathcal{M}_{\mathcal{R}}(z)$ and $\mathcal{L}_{\mathcal{R}}(z)$ are themselves databases. The next axiom applies this notation, stating general circumstances under which a set of rankings should lead to comparability of eventualities given a database. The notation **0** stands for the all-zeros database.

Exact Independence of Relevance: Let x, y be eventualities, K a database, and $\mathcal{R}$ a set of rankings. If

$$\mathcal{M}_{\mathcal{R}}(x) - \mathcal{L}_{\mathcal{R}}(x) = K \ , \tag{2}$$

$$\mathcal{L}_{\mathcal{R}}(y) - \mathcal{M}_{\mathcal{R}}(y) = K \ , \text{ and} \tag{3}$$

$$\mathcal{M}_{\mathcal{R}}(z) - \mathcal{L}_{\mathcal{R}}(z) = \mathbf{0} \ , \quad \text{for every } z \neq x, y \ , \tag{4}$$

then $x \succsim_K y$.

Note that in the special case where all the relations in $\mathcal{R}$ apply to the same two eventualities, Exact Independence of Relevance is reduced to Strong Combination, which is an expression of independence across records. On the other hand, if the relations that appear in $\mathcal{R}$ all refer to the same database, then Exact Independence of Relevance reduces to transitivity, which is an expression of independence across eventualities. In general, Exact Independence of Relevance allows both the databases and the eventualities to vary, indicating that both types of independence hold concurrently. The following discussion explains the concepts behind this axiom and elucidates how they comply with the two notions of independence, both across records and across eventualities.

By equation (2), the aggregated database on which x is considered **more** likely than other eventualities in $\mathcal{R}$, is the sum of K and the database on which x is considered **less** likely than other eventualities in $\mathcal{R}$. Following the logic that underlies Strong Combination, the records in K must be responsible for the evaluation of x as more likely. It thus follows that K contains evidence to corroborate the evaluation of x as more likely relative to the other eventualities in $\mathcal{R}$, while equation (3) analogously provides evidence in K to corroborate the evaluation of y as less likely relative to the other eventualities in $\mathcal{R}$.

Nevertheless, it is undesirable to conclude that x is more likely than y on K based on equations (2) and (3) alone. This is clearly demonstrated in the following example in which $x \succsim_K z$ and $w \succsim_K y$ are known. It is purely speculative to infer that necessarily $x \succsim_K y$, which would follow if only conditions (2) and (3) were to be imposed, since it is perfectly possible that $w \succsim_K y \succsim_K x \succsim_K z$. Concluding that $x \succsim_K y$ becomes reasonable if certain additional relations are known, such as Conditions (4). The most straightforward way in which Conditions (4) could be satisfied, in this example, is if it were known that $z \succsim_K w$; as transitivity would imply that $x \succsim_K y$. In this case, z and w are utilized as references for the purposes of comparing x and y. Eventuality x is more likely than eventuality z, which itself is more likely than eventuality w. At the same time eventuality y is less likely than eventuality w on K. All of which taken together would lead the evaluator to conclude that x is at least as likely as y on K.

However, Conditions (4) can also be satisfied under alternative, more complex, sets of relations. For example, they would be satisfied if, instead of $z \succsim_K w$, it were known that $z \succsim_{K+J} y$, $x \succsim_{K+J} w$, $y \succsim_J x$, and $w \succsim_J z$. When these relations are taken into account, Conditions (2) and (3) are satisfied for $2K$, which is preference-wise identical to K, and Conditions (4) are satisfied having that, $\mathcal{L}_{\mathcal{R}}(z) = \mathcal{M}_{\mathcal{R}}(z) = \mathcal{L}_{\mathcal{R}}(w) = \mathcal{M}_{\mathcal{R}}(w) = K + J$. Independence across records and across eventualities lead to the conclusion that $x \succsim_K y$: if $y \succsim_{K+J} x$ then by transitivity $z \succsim_{K+J} w$, and by Strong Combination $z \succsim_K w$, which again by transitivity implies that $x \succsim_K y$. If instead $x \succsim_{K+J} y$ then Strong Combination immediately implies that $x \succsim_K y$. In this example once again, z and w serve as references, only this time, not in relation to a single database but to an array of databases and eventualities. Conditions (4) guarantee that the aforementioned eventualities can serve as a reference for the relation between x and y on K . These other eventualities, overall neutralize themselves, since they gain no support for or against them when compared to x and y, as x gains support for being more likely and y less likely than these eventualities on K.

The axiom makes use of the two notions of independence, across records and across eventualities, in the interpretation of $\mathcal{M}_{\mathcal{R}}(\cdot)$ and $\mathcal{L}_{\mathcal{R}}(\cdot)$. Evidence that an eventuality is more likely, as well as evidence that it is less likely, is aggregated across databases, in order to obtain both the 'more likely' and the 'less likely' databases for the eventuality, $\mathcal{M}_{\mathcal{R}}(\cdot)$ and $\mathcal{L}_{\mathcal{R}}(\cdot)$. The resulting databases are independent of the order and manner in which they were aggregated, thus expressing independence of relevance across records. In addition, $\mathcal{M}_{\mathcal{R}}(\cdot)$ and $\mathcal{L}_{\mathcal{R}}(\cdot)$ are formed on the basis of comparisons of a particular eventuality to various other eventualities, reflecting the idea that the corroboration a record lends to the evaluation of an eventuality is independent of the eventualities to which it is being compared.[8]

2.2.4 Independence of Relevance

Exact Independence of Relevance is generalized to express the requirement that independence of relevance across records and eventualities be a continuous property. Namely, if there are rankings that suitably approximate the conditions that appear in Exact Independence of Relevance, then again it must be that $x \succsim_K y$.

A3. Independence of Relevance: Let x, y be eventualities, and K a database. Suppose there exists a sequence of pairs $(\mathcal{R}_i, n_i)$ where each $\mathcal{R}_i$ is a set of rankings and $n_i \in \mathbb{N}$, such that (convergence is point-wise, case-by-case)

$$\begin{aligned} \left(\mathcal{M}_{\mathcal{R}_i}(x) - \mathcal{L}_{\mathcal{R}_i}(x)\right)/n_i &\rightarrow K \\ \left(\mathcal{L}_{\mathcal{R}_i}(y) - \mathcal{M}_{\mathcal{R}_i}(y)\right)/n_i &\rightarrow K \\ \left(\mathcal{M}_{\mathcal{R}_i}(z) - \mathcal{L}_{\mathcal{R}_i}(z)\right)/n_i &\rightarrow \mathbf{0}\,, \quad \text{for every } z \neq x, y\,. \end{aligned}$$

Then $x \succsim_K y$.

These conditions are meant to capture closeness of $\mathcal{M}_{\mathcal{R}_i}(x) - \mathcal{L}_{\mathcal{R}_i}(x)$ and $\mathcal{L}_{\mathcal{R}_i}(y) - \mathcal{M}_{\mathcal{R}_i}(y)$ to $n_i K$,[9] and of $\mathcal{M}_{\mathcal{R}_i}(z) - \mathcal{L}_{\mathcal{R}_i}(z)$ to the all-zeros database. The conclusion

[8] Exact Independence of Relevance is closely connected to the finite cancellation condition that was first introduced by Kraft, Pratt and Seidenberg (1959) to derive a representation of a subjective probability. The primitives over which the finite cancellation condition is defined are events rather than databases as in our paper. Nevertheless, the two conditions impose analogous strong additivity requirements, which serve to derive an additive representation.

[9] Any relation that holds for K holds for any replication of K and vice versa, due to Homogeneity,

is then the same as in the exact version of Independence of Relevance.

2.2.5 Axioms for the Characterization

The axioms that are finally imposed in the model are Reflexivity (A1), Replication-Invariance of Incomparability (A2), and Independence of Relevance (A3). The next remark summarizes attributes that follow from these axioms, starting from transitivity and Strong Combination, that were already discussed. The next attribute is Homogeneity, which states that likelihoods of eventualities are comparable given a database if and only if they are comparable given its replications. Homogeneity shows that incomparability in our model is unrelated to the amount of available data, in that incomparability will not be resolved by simply having more data, nor by having less. This property is compatible with conflicting interpretations being a source of incomparability, as any conflicting interpretations that obtain for a database obtain for its replications, and vice versa. The last attribute in the remark, termed Impartiality, asserts that on the empty database (the all-zeros database) all eventualities are comparable and deemed equally likely.[10] Impartiality is yet another demonstration that incomparability is not a function of the amount of data but rather of conflicts in their analysis – no available data whatsoever generates no conflicting interpretations of data, leading to comparability (in the form of indifference) of all eventualities.

Remark 1. For every three eventualities x, y, z and databases I and J, Reflexivity, Replication-Invariance of Incomparability and Independence of Relevance imply:

(a) Transitivity: If $x \succsim_I y$ and $y \succsim_I z$, then $x \succsim_I z$.

(b) Strong Combination:

If $x \succsim_I y$ and $x \succsim_J y$, then $x \succsim_{I+J} y$.

If $x \succsim_{I+J} y$ and $y \succsim_J x$, then $x \succsim_I y$.

a property that follows from A2 and A3 and will be discussed shortly. These conditions therefore reflect that the differences preference-wise converge to K. Thus, for example, we view the sequence of databases $(10n + 1, 3n + 2)$ as approaching, preference-wise, the database $(10, 3)$.

[10] In particular, Impartiality excludes some a-priori structures of the eventualities considered. For instance, eventualities cannot be nested.

(c) Homogeneity: For any $n \in \mathbb{N}$, $x \succsim_I y$ if and only if $x \succsim_{nI} y$.

(d) Impartiality: $x \succsim_0 y$, where 0 denotes the all-zeros database.

3 Results

3.1 Main Result

The conclusion of this inquiry is that assumptions A1-A3 are equivalent to the representation in (1). As the domain of the relevance-weighting functions is $\mathbb{X} \times \mathbb{C}$, they are referred to as matrices throughout the remainder of the paper.

Theorem 1. *The following statements are equivalent:*

(i) The relations $\{\succsim_I\}_{I \in \mathbb{J}}$ satisfy assumptions A1-A3.

(ii) There exists a nonempty set V of matrices $v : \mathbb{X} \times \mathbb{C} \longrightarrow \mathbb{R}$, such that for any two eventualities x, y and database I,

$$x \succsim_I y \iff \sum_{c \in \mathbb{C}} v(x,c)I(c) \geq \sum_{c \in \mathbb{C}} v(y,c)I(c), \quad \textit{for every } v \in V \ . \tag{5}$$

With this representation each matrix $v \in V$ can be viewed as a conceivable interpretation of the data. The likelihood of eventuality x being greater than that of y is accepted only when every possible interpretation of the data leads to that conclusion. If, however, x is evaluated as more likely than y according to one interpretation, and less likely according to another interpretation, then no ranking of the likelihoods of the two eventualities is possible.

Example 2. In order to illustrate the evaluation process described by our theorem let us return to the medical example presented in the introduction. Assume that the doctor has the following set of data on past patients:

Fatigue	Fever	Cough	Outcome	Num. of Records
No	Yes	Yes	Illness A	150
Yes	No	Yes	Illness A	4
No	Yes	No	Illness A	60
Yes	No	No	Illness A	5
No	Yes	Yes	Illness B	40
No	Yes	No	Illness B	30
Yes	No	No	Illness B	40

Now suppose that a patient suffering from a cough and fatigue, but without fever, asks the doctor for a diagnosis. Considering illnesses A and B, the doctor may be inclined to diagnose the current patient with illness A as the proportion of A patients in the entire population is approximately 2/3. However, the fraction of A patients varies dramatically within different subgroups of the population, therefore the degree of relevance of these past cases to this specific patient is crucial.

In the above database there are only four cases identical to the current case. These cases are the most relevant to the case at hand, so the doctor would surely take them into account when comparing the likelihoods of illnesses A and B. For example, this corresponds to assigning, for $x \in \{\text{illness A}, \text{illness B}\}$ and every considered relevance-weighting function v, $v(x,c) = 1$ when $c = (\text{fatigue yes}, \text{fever no}, \text{cough yes}, x)$. However, the doctor may want to take into account less relevant cases. If the doctor were to consider in addition only past cases of patients who had a cough, he or she would conclude that illness A is more likely, as it is much more frequent than B in that group. For instance, this corresponds to assigning $v(x,c) = 0.5$, when $c = (\text{fatigue no}, \cdot, \text{cough yes}, x)$ or $c = (\text{fatigue yes}, \text{fever yes}, \text{cough yes}, x)$, $v(x,c) = 1$ when $c = (\text{fatigue yes}, \text{fever no}, \text{cough yes}, x)$, and $v(x,c) = 0$ otherwise. If, instead, the doctor were to consider, in addition to identical cases, only past cases of patients who suffered from fatigue, the doctor would come to the opposite conclusion, as illness B within this group is much more frequent than A. This corresponds to assigning $v(x,c) = 0.5$ when $c = (\text{fatigue yes}, \cdot, \text{cough no}, x)$ or $c = (\text{fatigue yes}, \text{fever yes}, \text{cough yes}, x)$, $v(x,c) = 1$ when $c = (, \text{fatigue yes}, \text{fever no}, \text{cough yes}, x)$

and $v(x, c) = 0$ otherwise.

It is precisely this contradiction in the conclusions that makes the doctor unsure which illness is more likely in the current case. Importantly, such uncertainty does not exist when considering a patient suffering from a cough and a fever (but no fatigue), since there are many past cases of patients that suffered from those exact symptoms that indicate that illness A is more likely than B. Likewise, gathering enough observations of patients with a cough and fatigue will settle the comparison for the case under consideration.

In principle, the accumulation of observations that are clearly relevant to the question at hand will typically resolve the comparison between eventualities. When observations whose characteristics are identical (or almost identical) to those of the current case are accumulated then, assuming that all the employed relevance-weighting functions put very high weight on such observations relative to more distant ones, a comparison would become valid.

Another aspect that is fundamental to case-based reasoning is that weights v are held constant across databases, precluding processes that entail learning of relevance weights. The assumption made is that a specific set of relevance-weights is applied in all comparisons of eventualities, rather than this set itself being updated constantly as observations accumulate. The set of weights may be the result of a learning process, but this process is exogenous to our model. The evaluator therefore entertains a fixed set of interpretations that are not altered during the evaluation process. For further discussion of this point see GS on *second-order induction.*

In comparison with the GS model, instead of our assumption of Independence of Relevance (which in itself includes a form of continuity) they impose Combination and Continuity, add Completeness, and supplement the conditions with a richness assumption termed Diversity. For the sake of convenience we restate the Continuity and Diversity axioms that appear in GS, and their result below. Continuity is formulated with negation of rankings, which translates, for complete relations, to strict preferences.

GS Continuity: For any databases I, J and eventualities x, y, if $\neg(x \succsim_I y)$ then there exists $n \in \mathbb{N}$ such that $\neg(x \succsim_{nI+J} y)$.

GS Diversity: For every list of four distinct eventualities (x, y, z, w) there exists a database I such that $x \succ_I y \succ_I z \succ_I w$. If $|\mathbb{X}| < 4$, then for any strict ordering of the elements of $\mathbb{X}$ there exists a database I such that $\succ_I$ is that ordering.

Theorem 2 (GS, 2003)**.** *The following statements are equivalent:*

(i) The relations $\{\succsim_I\}_{I \in \mathbb{J}}$ satisfy Completeness and Transitivity, Combination, Continuity and Diversity.

(ii) There exists a matrix $v : \mathbb{X} \times \mathbb{C} \longrightarrow \mathbb{R}$, such that for any two eventualities x, y and database I,

$$x \succsim_I y \iff \sum_{c \in \mathbb{C}} v(x, c)I(c) \geq \sum_{c \in \mathbb{C}} v(y, c)I(c).$$

Furthermore, v is diversified,[11] *and is unique in the sense that v and u both represent $\{\succsim_I\}_{I \in \mathbb{J}}$ as above iff there is a scalar $\lambda > 0$ and a matrix $\beta : \mathbb{X} \times \mathbb{C} \longrightarrow \mathbb{R}$ with identical rows (i.e., with constant columns) such that $u = \lambda v + \beta$.*

Diversity is a strong assumption in our context, since it requires the evaluator to be able to determine many comparisons, and thus confines the type of incomplete relations that can be described. In our work we therefore employ a strong form of additivity, generalizing Combination, instead of imposing Diversity.

3.1.1 *Uniqueness of the Representing Set* V

The set of matrices V in our theorem need not be unique. First, if V represents the relations $\{\succsim_I\}_{I \in \mathbb{J}}$, then so does the closed convex cone generated by V, and second, it is possible to shift the matrices in V by adding any matrix with identical rows and still obtain a representation of the same relations. These issues can be resolved by normalizing some row of each matrix in V, for example, by fixing the last row in each

[11] If $|\mathbb{X}| \geq 4$, v is diversified if there are no distinct four eventualities x, y, z, w and real numbers $\lambda, \mu, \theta,\ \lambda + \mu + \theta = 1$, such that $v(x, \cdot) \leq \lambda v(y, \cdot) + \mu v(z, \cdot) + \theta v(w, \cdot)$. If $|\mathbb{X}| < 4$, v is diversified if no row in v is dominated by an affine combination of the others.

matrix to equal an all-zeros row. Yet, even if all matrices in V are normalized, and only closed convex cones are considered, uniqueness is not guaranteed. The following example demonstrates this point.

Example 3. Let $\mathbb{X} = \{x, y, z\}$, $\mathbb{C} = \{1, 2, 3\}$, and define V to be the closed convex cone generated by the following three matrices:

$$v_1 = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad v_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad v_3 = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \quad .$$

The relations obtained are $x \succsim_I y \;\Leftrightarrow\; I(1) \geq 3I(2)$, $x \succsim_I z \;\Leftrightarrow\; I(1) \geq 3I(3)$, and $y \succsim_I z \;\Leftrightarrow\; I(2) \geq 3I(3)$, for any database I. Now consider the matrix

$$v_4 = \begin{pmatrix} 5/12 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} .$$

If the matrices above are supplemented with v_4 the relations do not change in any way, in spite of v_4 not being contained in V. In other words, if W is the closed convex cone generated by $v_1, \ldots, v_4$, then W and V are two distinct sets, each representing the same relations $\{\succsim_I\}_{I \in \mathbb{J}}$. Notice that the relations in this example satisfy the Diversity condition of GS. For complete relations Diversity guarantees uniqueness of relevance weights. However, the example demonstrates that this no longer holds for incomplete relations.

Without uniqueness, a 'standard' representing set of matrices is the maximal representing set w.r.t. set inclusion. This is the union of all representing sets, which is a closed convex cone and unique by definition. The maximal set has the advantage of containing all possible values of relevance-weights. If, after employing a consensus rule over various interpretations of data as a first stage of comparison, a further examination of completion of relations is undertaken by employing a different rule (e.g. a minimum-weight rule), then no relevance-weights are eliminated *a priori*. The second rule then relies on the full range of interpretations that are compatible with the initial incomplete relations.

3.1.2 *Continuity Axiom*

One may wonder whether our representation can be derived using a more standard form of continuity, such as the GS continuity stated above. That is, suppose that the continuous version of Independence of Relevance, namely A3, is replaced by Exact Independence of Relevance, together with GS Continuity. The following example shows, however, that this modified set of axioms is weaker than A1-A3, as it is insufficient to derive the representation (1).

Example 4. Let $\mathbb{X} = \{x, y, z\}$, $\mathbb{C} = \{1, 2\}$. Suppose that Impartiality holds, and that for every I, $\succsim_I$ is reflexive. In addition, assume the following relations (and only those) are satisfied:

(i) For any database I such that $I(1) \geq I(2)$, $x \succsim_I y$.

(ii) For any database J such that $J(1) \leq J(2)$ and $\sqrt{2} \cdot J(1) > J(2)$, $x \succsim_J z$.

(iii) For any database Q such that $\sqrt{2} \cdot Q(1) < Q(2)$, $z \succsim_Q y$.

It is next shown that for these relations a representation as in our theorem does not exist. To see this, suppose on the contrary that there is a representing set V of matrices in the sense of (ii) of Theorem 1. For relations (ii) and (iii) to hold, every $v \in V$ should satisfy:

$$\forall (k, m) \in \mathbb{J} \text{ s.t. } k \leq m < \sqrt{2}k \ , \ k[v(x, 1) - v(z, 1)] + m[v(x, 2) - v(z, 2)] \geq 0 \ ,$$
$$\forall (k, m) \in \mathbb{J} \text{ s.t. } m > \sqrt{2}k \ , \ k[v(z, 1) - v(y, 1)] + m[v(z, 2) - v(y, 2)] \geq 0 \ .$$

These inequalities hold true when k is substituted by any $\alpha > 0$, and m by $\sqrt{2}\alpha$, which together imply that $\alpha[v(x, 1) - v(y, 1)] + \sqrt{2}\alpha[v(x, 2) - v(y, 2)] \geq 0$. Moreover, for every $\beta \geq \gamma > 0$, $\beta[v(x,1) - v(y,1)] + \gamma[v(x,2) - v(y,2)] \geq 0$, since by (i), $k[v(x,1) - v(y,1)] + m[v(x,2) - v(y,2)] \geq 0$ for every $k \geq m$. Hence for $\alpha, \beta, \gamma > 0$ for which $\beta \geq \gamma$, $(\alpha+\beta)[v(x,1) - v(y,1)] + (\sqrt{2}\alpha + \gamma)[v(x,2) - v(y,2)] \geq 0$. Given any k and m, by letting $\alpha = \frac{m-k}{\sqrt{2}-1}$ and $\beta = \gamma = \frac{\sqrt{2}k-m}{\sqrt{2}-1}$ we obtain $k[v(x,1) - v(y,1)] + m[v(x,2) - v(y,2)] \geq 0$.

Consider a database $K = (k, m)$ with $k < m < \sqrt{2}k$. As the inequalities are satisfied for every $v \in V$, it is implied that $x \succsim_K y$, in contrast to the supposition that x and y are incomparable on K. We thus conclude that the relations defined above do not admit a representation as in our theorem.

On the other hand, the relations as defined do satisfy assumptions A1, A2, Exact Independence of Relevance, and GS Continuity. This is proved in appendix A. Exact Independence of Relevance together with GS Continuity are therefore weaker than A3, as Under A1-A3 a representation as in Theorem 1 obtains (in appendix A it is also shown directly that A3 is violated in the example). Intuitively, what drives the proof is that there are no databases on the real line $\{(x_1, x_2) : \sqrt{2} \cdot x_1 = x_2\}$, since x_1 and x_2 cannot both be integers.[12] Exact Independence of Relevance therefore has no bite there.

3.2 Complete Relations

If completeness is assumed for relations $\{\succsim_I\}_{I \in \mathbb{J}}$, then a conclusion similar to GS can be obtained.

A4. Completeness: For any two eventualities x, y and database I, either $x \succsim_I y$ or $y \succsim_I x$.

Proposition 3. *The following statements are equivalent:*

(i) The relations $\{\succsim_I\}_{I \in \mathbb{J}}$ satisfy assumptions A3 and A4.

[12]Technically speaking, if the domain of databases were real-valued, i.e., the databases were $\mathbb{R}_+^{\mathbb{C}}$, then A1, A2, Exact Independence of Relevance, and GS Continuity would actually be sufficient to derive the representation (1) (though the proof is not trivial).

(ii) There exists a matrix $v : \mathbb{X} \times \mathbb{C} \longrightarrow \mathbb{R}$*, such that for any two eventualities* x, y *and database* I*,*

$$x \succsim_I y \iff \sum_{c \in \mathbb{C}} v(x,c)I(c) \geq \sum_{c \in \mathbb{C}} v(y,c)I(c) \quad . \tag{6}$$

There are two notable differences between the representation obtained here and that of the GS model, stated above as Theorem 2. First, the assumptions used here imply a broader range of relations, as the relations are not required to be diverse, and correspondingly the representing matrix is not necessarily diversified. On the other hand, while in the GS model the representing matrix is essentially unique (up to multiplication by a positive constant and shift by a matrix with identical rows), Proposition 3 does not provide such uniqueness. The following is an example of a family of complete relations which satisfy (i) of Proposition 3, yet admit two different representations by two distinct matrices.

Example 5. Let $\mathbb{X} = \{x, y, z\}$, $\mathbb{C} = \{1, 2, 3\}$, and

$$v = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0.5 & 0.5 & 2.5 \end{pmatrix} .$$

The rankings defined by v through the representation (6) are: if $I(1) + I(3) \geq I(2)$, then $x \succsim_I z \succsim_I y$, otherwise $y \succsim_I z \succsim_I x$. This representation, however, is not unique. The same rankings are induced, for instance, by the matrix:

$$w = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0.25 & 0.75 & 2.25 \end{pmatrix} .$$

The difference between the representation of Proposition 3 and that of GS stems from the fact that the axioms of Combination and Diversity used in the GS model are replaced by Independence of Relevance. As discussed in Subsection 2.2, Independence of Relevance generalizes and strengthens Combination and Transitivity. In doing so, it makes Diversity unnecessary. However, Independence of Relevance, as mentioned before, does not yield a unique representing matrix. Note that in the example above Diversity is not satisfied, as eventuality z is always ranked between x and y. This example therefore cannot be accommodated in the GS model.

4 Conclusion

In the present work we describe an evaluation process that compares likelihoods of eventualities based on past data. A central feature of the model is that different plausible interpretations of the data may arise when the evaluator is uncertain as to how relevant the data are to the question at hand. The evaluator is able to compare the likelihoods of eventualities just as long as all interpretations of the data lead to the same conclusion, while refraining from making a comparison when opposite conclusions obtain.

These difficulties may be resolved by gathering more data for which there is less doubt regarding their applicability, such as records of patients whose characteristics more closely resemble those of the current case (e.g. more patients that suffer from cough and fatigue in the medical example in subsection 3.1). Unfortunately, such data may not always be available, leaving comparisons undetermined.

The medical examples in the introduction demonstrate that incompleteness in likelihood evaluations should not be deemed 'irrational' when the relevance of available data is doubtful. Furthermore, when an evaluator and a decision maker are two separate entities, the possibility to report non-determinability of likelihood comparisons allows the decision maker to incorporate additional considerations.

5 Proofs

5.1 Proof of Remark 1

For transitivity, apply A3 to the rankings $x \succsim_I y$, $y \succsim_I z$, which indicate that database I corroborates a (weak) ranking of x above z. For Combination, apply A3 to the rankings $x \succsim_I y,\ x \succsim_J y$, which indicate that database $I+J$ corroborates a ranking of x above y. In order to see that one direction of Homogeneity holds, apply A3 to n copies of the ranking $x \succsim_I y$, which indicate that database nI corroborates a ranking of x above y. In the other direction apply axiom A2. Last, Impartiality is seen to be satisfied by applying A3 to the rankings $x \succsim_I x,\ y \succsim_I y$, which hold true by Reflexivity (for any database I). These rankings indicate that the all-zeros database corroborates a ranking of x above y, as well as y over x.

5.2 Proof of Theorem 1

5.2.1 *Proof of the Direction (i)⇒(ii)*

The proof of this direction, as well as the proof of Proposition 3, use definitions and arguments from Ashkenazi and Lehrer (2001). Similar techniques can also be found in Dubra and Ok (2002).

We extend the relations $\{\succsim_I\}_{I\in\mathbb{J}}$ to rational-valued databases, namely to relations $\{\succsim_I\}_{I\in\mathbb{Q}_+^{\mathbb{C}}}$, in the following manner: For a rational-valued database $I \in \mathbb{Q}_+^{\mathbb{C}}$ let $k \in \mathbb{N}$ be such that $kI \in \mathbb{Z}_+^{\mathbb{C}}$, and define $\succsim_I = \succsim_{kI}$. By Homogeneity (see Remark 1) the extension is well defined, and satisfies $\succsim_I = \succsim_{qI}$ for every $I \in \mathbb{Q}_+^{\mathbb{C}}$ and $q \in \mathbb{Q}$, $q > 0$. Assumptions A1-A2 as well as Exact Independence of Relevance immediately carry through to the extended relations and will thus be employed without further mention. Axiom A3, when used in course of the proof, will explicitly be applied to integer-valued databases.

Consider the Euclidean space $\mathbb{R}^{\mathbb{X}\times\mathbb{C}}$ as consisting of vectors with blocks, where each block corresponds to an eventuality, and within each block coordinates correspond to record types. For $\varphi \in \mathbb{R}^{\mathbb{X}\times\mathbb{C}}$, φ_x denotes the block matching eventuality x, which is a vector in $\mathbb{R}^{\mathbb{C}}$. For $v \in \mathbb{R}^{\mathbb{X}\times\mathbb{C}}$, eventuality x and record type c, $v(x, c)$ denotes the entry of v which belongs to block x, and inside block x, to record type c. For any two eventualities $x, y \in \mathbb{X}$ and a rational-valued database I, the vector $\varphi(I, x, y)$ denotes the vector in $\mathbb{R}^{\mathbb{X}\times\mathbb{C}}$ for which the block corresponding to eventuality x is I, the block corresponding to eventuality y is $-I$, and those corresponding to other eventualities are all zero. Consider $\mathcal{B} = conv\{\varphi(I, x, y) \mid x, y \in \mathbb{X},\ I \in \mathbb{Q}_+^{\mathbb{C}}\} \subset \mathbb{R}^{\mathbb{X}\times\mathbb{C}}$, with its natural relative topology generated by the Euclidean topology over $\mathbb{R}^{\mathbb{X}\times\mathbb{C}}$ (namely with the topology of sets $B \cap \mathcal{B}$, for open sets B in the Euclidean topology of $\mathbb{R}^{\mathbb{X}\times\mathbb{C}}$). All steps in the proof from now on are conducted in this relative topology.

Define $E = cl(conv\{\varphi(I,x,y) \mid x \succsim_I y,\ x,y \in X,\ I \in \mathbb{Q}_+^{\mathbb{C}}\})$. That is, E is the closed convex hull generated by all vectors in $\mathcal{B}$ that indicate ranking. As Homogeneity and Impartiality are satisfied, E is a closed (in the relative topology) convex cone with vertex zero. By its definition, E contains all vectors that indicate ranking. That is, if $x \succsim_I y$ then $\varphi(I,x,y) \in E$. The following claim shows that the opposite is also true, namely that if a vector of the form $\varphi(I,x,y)$ is contained in E, then $x \succsim_I y$.

Claim 4. *If $\varphi(I,x,y) \in E$ then $x \succsim_I y$.*

Proof. If $\varphi(I,x,y) \in E$ then so is $\varphi(iI,x,y)$ for iI that is an integer-valued database. For any such $\varphi(iI,x,y)$ there exists a sequence of points converging to $\varphi(iI,x,y)$, $\varphi^n = \sum_{t=1}^{T_n} \lambda_t^n \varphi(I_t^n, m_t^n, \ell_t^n)$, where $\lambda_t^n > 0$ and $m_t^n \succsim_{I_t^n} \ell_t^n$.

By denseness of the rational numbers in the reals there must also be a sequence $q^n = \sum_{t=1}^{T_n} r_t^n \varphi(I_t^n, m_t^n, \ell_t^n) = \sum_{t=1}^{T_n} \varphi(r_t^n I_t^n, m_t^n, \ell_t^n)$, with $r_t^n > 0$ rational numbers, that also converges to $\varphi(iI,x,y)$.

By the structure of $\varphi(iI,x,y)$, if q^n converges to $\varphi(iI,x,y)$, then

$$\begin{aligned}
&\sum_{t:m_t^n=x} r_t^n I_t^n - \sum_{t:\ell_t^n=x} r_t^n I_t^n \to iI \\
&\sum_{t:\ell_t^n=y} r_t^n I_t^n - \sum_{t:m_t^n=y} r_t^n I_t^n \to iI \\
&\sum_{t:m_t^n=z} r_t^n I_t^n - \sum_{t:\ell_t^n=z} r_t^n I_t^n \to \mathbf{0}\,, \quad \text{for every } z \neq x,y\ .
\end{aligned}$$

For any n, let d^n be an integer such that $J_t^n := d^n r_t^n I_t^n$ is an integer-valued database for all t. Then still $m_t^n \succsim_{J_t^n} \ell_t^n$, and

$$\left(\sum_{t:m_t^n=x} J_t^n - \sum_{t:\ell_t^n=x} J_t^n\right)/d^n \to iI$$

$$\left(\sum_{t:\ell_t^n=y} J_t^n - \sum_{t:m_t^n=y} J_t^n\right)/d^n \to iI$$

$$\left(\sum_{t:m_t^n=z} J_t^n - \sum_{t:\ell_t^n=z} J_t^n\right)/d^n \to \mathbf{0}, \quad \text{for every } z \neq x, y \ .$$

Axiom A3 implies that $x \succsim_{iI} y$, and thus also $x \succsim_I y$. ■

We thus conclude that E contains exactly those vectors $\varphi(I, x, y)$ for which $x \succsim_I y$.

Define $V = \{v \in \mathbb{R}^{\mathbb{X}\times\mathbb{C}} \mid v \cdot \varphi \geq 0 \text{ for all } \varphi \in E\}$ (where $v \cdot \varphi$ denotes the inner product of v and φ). The set V is not empty, since it contains zero. If v_1 and v_2 are in V then so is $\alpha v_1 + \beta v_2$ for $\alpha, \beta \geq 0$, and if $v_n \to_{n\to\infty} v$ for $v_n \in V$, then $v \in V$. Hence V is a closed convex cone, with vertex zero.

By the definition of V and Claim 4, if $x \succsim_I y$ then $\sum_{c\in\mathbb{C}} v(x, c)I(c) \geq \sum_{c\in\mathbb{C}} v(y, c)I(c)$ for every $v \in V$. In the other direction, suppose that $\neg(x \succsim_I y)$, that is, $\varphi(I, x, y) \notin E$. Since E is closed and convex, then by a separation theorem there exists a vector $v \in \mathbb{R}^{\mathbb{X}\times\mathbb{C}}$ separating E and $\varphi(I, x, y)$. By Impartiality the zero vector belongs to E, and by Homogeneity if $\varphi(I, x, y) \notin E$ then $\varphi(qI, x, y) = q\varphi(I, x, y) \notin E$ for any $q \in \mathbb{Q}$. Hence the separating scalar is zero, that is, $v \cdot \psi \geq 0 > v \cdot \varphi(I, x, y)$, for every $\psi \in E$. In other words, there exists $v \in V$ such that $\sum_{c\in\mathbb{C}} v(x, c)I(c) < \sum_{c\in\mathbb{C}} v(y, c)I(c)$. Thus it is established that there exists a set $V \subseteq \mathbb{R}^{\mathbb{X}\times\mathbb{C}}$, such that for every pair of eventualities x, y and database I,

$$x \succsim_I y \quad \text{if and only if} \quad \sum_{c\in\mathbb{C}} v(x, c)I(c) \geq \sum_{c\in\mathbb{C}} v(y, c)I(c) \text{ for all } v \in V \ .$$

5.2.2 *Proof of the Direction (ii)⇒(i)*

Suppose that the relations $\{\succsim_I\}_{I\in\mathbb{J}}$ can be represented by (ii) of Theorem 1. It is immediate to see that Reflexivity (A1) is satisfied. Assumption A2 is proved by multiplying both sides of the inequality by the same constant.

For Independence of Relevance (A3), let x, y be eventualities and K a database, such that there is a sequence of pairs $(\mathcal{R}^i, n^i)$, where each $\mathcal{R}^i$ is a set of rankings $\mathcal{R}^i = \{m_1^i \succsim_{J_1^i} \ell_1^i, \ldots, m_r^i \succsim_{J_r^i} \ell_r^i\}$ and

$$\left(\sum_{t:m_t^i=x} J_t^i - \sum_{t:\ell^i{}_t=x} J_t^i\right)/n^i \;\;\to\;\; K \;,$$
$$\left(\sum_{t:\ell_t^i=y} J_t^i - \sum_{t:m_t^i=y} J_t^i\right)/n^i \;\;\to\;\; K \;,$$
$$\left(\sum_{t:m_t^i=z} J_t^i - \sum_{t:\ell^i{}_t=z} J_t^i\right)/n^i \;\;\to\;\; \mathbf{0} \;, \quad \text{for every } z \neq x, y \;.$$

Let us fix, for the moment, an index i and suppress the i superscript, for convenience. By summing over all the inequalities implied by the relations,

$$\sum_{t=1}^{r}\left[\sum_{c\in\mathbb{C}} v(m_t, c)J_t(c) - \sum_{c\in\mathbb{C}} v(\ell_t, c)J_t(c)\right] =: \alpha_v \geq 0 \;, \text{ for every } v \in V \;.$$

After rearranging the addends in the above expression according to the eventualities involved, it is obtained that:

$$
\begin{aligned}
0 \leq \alpha_v &= \sum_{t:m_t=x}\sum_{c\in\mathbb{C}} v(x,c)J_t(c) - \sum_{t:\ell_t=x}\sum_{c\in\mathbb{C}} v(x,c)J_t(c) + \\
&\quad \sum_{t:m_t=y}\sum_{c\in\mathbb{C}} v(y,c)J_t(c) - \sum_{t:\ell_t=y}\sum_{c\in\mathbb{C}} v(y,c)J_t(c) + \\
&\quad \sum_{z\neq x,y}\left[\sum_{t:m_t=z}\sum_{c\in\mathbb{C}} v(z,c)J_t(c) - \sum_{t:\ell_t=z}\sum_{c\in\mathbb{C}} v(z,c)J_t(c)\right] \\
&= \sum_{c\in\mathbb{C}} v(x,c)\left(\sum_{t:m_t=x} J_t(c) - \sum_{t:\ell_t=x} J_t(c)\right) + \\
&\quad \sum_{c\in\mathbb{C}} v(y,c)\left(\sum_{t:m_t=y} J_t(c) - \sum_{t:\ell_t=y} J_t(c)\right) + \\
&\quad \sum_{z\neq x,y}\left[\sum_{c\in\mathbb{C}} v(z,c)\left(\sum_{t:m_t=z} J_t(c) - \sum_{t:\ell_t=z} J_t(c)\right)\right]
\end{aligned}
$$

Now, if we divide the above expression by n^i then the result converges, as $i \to \infty$, to

$$
\begin{aligned}
&\sum_{c\in\mathbb{C}} v(x,c)K(c) + \sum_{c\in\mathbb{C}} v(y,c)\left(-K(c)\right) + \sum_{z\neq x,y}\left[\sum_{c\in\mathbb{C}} v(z,c)\cdot 0\right] \\
&\qquad = \sum_{c\in\mathbb{C}} \left[v(x,c) - v(y,c)\right] K(c)\,.
\end{aligned}
$$

As $\alpha_v^i \geq 0$ for every i, then also $\alpha_v^i / n^i \geq 0$. Therefore, the limit is nonnegative as well, namely, $\sum_{c\in\mathbb{C}} \left[v(x,c) - v(y,c)\right] K(c) \geq 0$ for every $v \in V$, implying $x \succsim_K y$.

5.3 Proof of Proposition 3

First, note that Reflexivity (A1) follows from Completeness (A4). Replication-Invariance of Incomparability (A2) is implied by A4 and A3 as follows. Suppose $\neg(x \succsim_I y)$. By Completeness $y \succsim_I x$, hence A3 implies $y \succsim_{(n-1)I} x$. Now if $x \succsim_{nI} y$ then, by A3, we get $x \succsim_I y$, contrary to our assumption. Therefore, $\neg(x \succsim_{nI} y)$.

Thus, axioms A1-A3 hold and, by Theorem 1, there exists a set V of matrices $v : \mathbb{X} \times \mathbb{C} \longrightarrow \mathbb{R}$ that represents the relation $\succsim$ as in (5). We can write (5) succinctly as $x \succsim_I y \Longleftrightarrow v \cdot \varphi(I,x,y) \geq 0$, for every $v \in V$ ($\varphi(I,x,y)$ was defined in the proof of

Theorem 1).

Let $Z = \{ (I,x,y) \mid \neg(x \succsim_I y),\ x,y \in X,\ I \in \mathbb{J}\}$ be the set of all possible relations that do not hold. Since X is finite and $\mathbb{J}$ is countable, there are countably many possible relations, hence Z is countable too. Choose some order on the elements of Z, and if (I,x,y) is the i-th element of Z then let w_i be some matrix in V for which $w_i \cdot \varphi(I,x,y) < 0$. W.l.o.g. $|w_i(x,c)| \leq 1$ for every $x \in X, c \in \mathbb{C}$, otherwise multiply w_i by a small scalar.

Let $v_n = \sum_{i=1}^{n} w_i/2^i$. Since each w_i is bounded, the sequence v_n converges to some matrix $v : \mathbb{X} \times \mathbb{C} \longrightarrow \mathbb{R}$ (namely, for every $x \in X, c \in \mathbb{C}$, $v_n(x,c)$ converges to $v(x,c)$).[13]

To see that the representation (6) holds with this single v, first take I,x,y for which $x \succsim_I y$. Since for each $w_i \in V$, $w_i \cdot \varphi(I,x,y) \geq 0$, then $v_n \cdot \varphi(I,x,y) \geq 0$ for every n, hence also $v \cdot \varphi(I,x,y) \geq 0$.

Next, take I,x,y for which $\neg(x \succsim_I y)$. Then (I,x,y) is the k-th element of Z, for some k. Denote $\delta = w_k \cdot \varphi(I,x,y) < 0$. On the other hand, $\neg(x \succsim_I y)$ implies $y \succsim_I x$, by Completeness. Therefore, $w_i \cdot \varphi(I,y,x) \geq 0$ for every i, and since $\varphi(I,x,y) = -\varphi(I,y,x)$, we get that $w_i \cdot \varphi(I,x,y) \leq 0$. Thus, for any $n \geq k$, $v_n \cdot \varphi(I,x,y) \leq \delta/2^k$. Therefore, also $v \cdot \varphi(I,x,y) \leq \delta/2^k < 0$.

As for the direction (ii)⇒(i), A3 follows from Theorem 1, since the representation (6) is a special case of (5). Completeness follows from there being a single representing matrix.

A Appendix: proof of Example 4

We show here that A1, A2, Exact Independence of Relevance, and GS Continuity obtain in Example 4, while A3 does not.

Reflexivity (A1) is explicitly stipulated. Homogeneity is directly implied by the conditions assumed, hence, in particular, Replication-Invariance of Incomparability (A2) obtains.

To see that Exact Independence of Relevance holds, let a,b,c denote the three

[13] If Z is empty, then $v = 0$.

distinct eventualities, K a database, and $\mathcal{R}$ a set of rankings, such that,

[a] $\mathcal{M}_{\mathcal{R}}(a) - \mathcal{L}_{\mathcal{R}}(a) = K$,

[b] $\mathcal{L}_{\mathcal{R}}(b) - \mathcal{M}_{\mathcal{R}}(b) = K$, and

[c] $\mathcal{M}_{\mathcal{R}}(c) - \mathcal{L}_{\mathcal{R}}(c) = \mathbf{0}$.

We should show that $a \succsim_K b$. First we note that when $\mathcal{R}$ only contains a single ranking then this is a mere tautology, as that single ranking must be $a \succsim_K b$ itself. W.l.o.g. $\mathcal{R}$ contains only rankings of cases (i)-(iii), since rankings implied by reflexivity or impartiality do not affect the calculus in [a], [b], and [c]. Suppose $\mathcal{R}$ contains rankings of case (i), $x \succsim_{I_1} y, \ldots, x \succsim_{I_m} y$. Case (i) is closed to addition, hence $x \succsim_{I_1+\ldots+I_m} y$ obtains. This last ranking can replace the previous ones in $\mathcal{R}$ without affecting the calculus. Therefore, w.l.o.g. $\mathcal{R}$ contains no more than one ranking of case (i); and similarly for (ii) and (iii).

Finally, suppose $\mathcal{R}$ contains more than one ranking, hence it involves more than one case out of (i)-(iii). Then $\mathcal{R}$ must contain at least one instance where x is ranked above y or z, and since x is never ranked below the others, it must be that $a = x$. Similarly, it must be that $b = y$. Therefore, $c = z$, implying that a ranking of case (ii) is counterbalanced by one of case (iii) in equation [c]. Yet this is impossible, as the databases of cases (ii) and (iii) are disjoint. Thus, we are left with the tautological form where $\mathcal{R}$ contains a single ranking.

To verify GS Continuity, we can check all pairs:
[1] $\neg(x \succsim_I y) \iff I(1) < I(2)$. For every such I and any database J, $(nI + J)(1) < (nI + J)(2)$, hence $\neg(x \succsim_{nI+J} y)$, for n large enough.
[2] $\neg(z \succsim_Q y) \iff \sqrt{2} \cdot Q(1) \geq Q(2)$. However, there exists no database for which $\sqrt{2} \cdot Q(1) = Q(2)$, since $Q(1), Q(2)$ are integers. Therefore, it is also true that $\neg(z \succsim_Q y) \iff \sqrt{2} \cdot Q(1) > Q(2)$, and the argument proceeds similarly to [1].
[3] $\neg(x \succsim_J z) \iff$ either $J(1) > J(2)$ or $\sqrt{2} \cdot J(1) < J(2)$, with the previous argument for either one of these two conditions.
The remaining pair rankings either obtain everywhere (e.g., $x \succsim_I x$ for every I) or only at $\mathbf{0}$ (e.g., $y \succsim_I x$ only when $I = (0, 0)$). Either way continuity follows immediately.

We now prove that our relations violate A3, by showing that for, e.g., $K = (10, 11)$ A3 would imply that $x \succsim_K y$, while in fact $\neg(x \succsim_K y)$. Denote $\alpha_n = (10\sqrt{2} - 11) \cdot n$, and define sequences of databases $I_n = (\lfloor \alpha_n \rfloor, \lfloor \alpha_n \rfloor)$, $J_n = (n, \lfloor \sqrt{2} \cdot n \rfloor)$, and $Q_n = (n, \lceil \sqrt{2}\, n \rceil)$. Then $x \succsim_{I_n} y$, $x \succsim_{J_n} z$, and $z \succsim_{Q_n} y$.

Denote $\nu_n = (\sqrt{2} - 1) \cdot n$. First, note that $\lim(Q_n - J_n)/\nu_n = (0, 0)$. Second, since $\alpha_n + n = (10 \cdot \sqrt{2} - 11 + 1) \cdot n = 10(\sqrt{2} - 1) \cdot n$, and $\alpha_n + \sqrt{2} \cdot n = 11(\sqrt{2} - 1) \cdot n$, then $\lim(I_n + J_n)/\nu_n = \lim(I_n + Q_n)/\nu_n = (10, 11) = K$.

Therefore, if we let $\mathcal{R}_n = \{x \succsim_{I_n} y,\, x \succsim_{J_n} z,\, z \succsim_{Q_n} y\}$ and $m_n = \lfloor \nu_n \rfloor$, then

$$
\begin{aligned}
\left(\mathcal{M}_{\mathcal{R}_n}(x) - \mathcal{L}_{\mathcal{R}_n}(x)\right)/m_n &\to K \\
\left(\mathcal{L}_{\mathcal{R}_n}(y) - \mathcal{M}_{\mathcal{R}_n}(y)\right)/m_n &\to K \\
\left(\mathcal{M}_{\mathcal{R}_n}(z) - \mathcal{L}_{\mathcal{R}_n}(z)\right)/m_n &\to \mathbf{0}
\end{aligned}
$$

and since $\neg(x \succsim_K y)$ we conclude that A3 is violated.

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