Worst-Case Expected Utility*

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Abstract

The paper presents a model in which a decision maker, having a preference relation over purely subjective acts, deviates minimally from the Subjective Expected Utility decision rule, exhibiting an uncertainty averse behavior *á*-la Schmeidler (1989). The resulting representation is as if the decision maker adds to the formulation of the problem one new state, representing the occurrence of some unforeseen event. Each Savage act is extended to the new, endogenous state by assigning this state with the worst consequence the act obtains on all other, primitive states. On the extended decision problem a Subjective Expected Utility rule is applied. The representation thus expresses the common practice of a 'worst-case scenario' assumption as means to cope with unforeseen contingencies. The model is a special case of the neo-additive capacities model of Chateauneuf, Eichberger and Grant.

Keywords: SEU, CEU, uncertainty aversion, neo-additive capacity, unforeseen contingencies

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1 Introduction

The paper presents an axiomatic model of decisions in which the decision maker is assumed to deviate minimally from the Subjective Expected Utility paradigm of Savage [19], and where all deviations are being driven by uncertainty aversion. The decision maker is modeled by a binary relation over Savage acts. The representation characterized is as if the decision maker adds a new state to the decision problem and extends each act to this new state. The new, endogenous state may be interpreted as 'some unforeseen event occurs'. In the representation caution is exhibited in that the consequence assumed for each act on the endogenous state is its worst consequence over all primitive (foreseen) states. On the extended decision problem, containing the new state, a Subjective Expected Utility rule is applied. The representation thus expresses the common practice of assuming a worst-case scenario in the face of the unknown, and the model is accordingly called Worst-Case Expected Utility (shortened to Worst-Case EU in the sequel). The model demonstrates that uncertainty averse behavior can emerge from a seemingly small deviation from Subjective Expected Utility.

The setup employed in the paper consists of a rich set of consequences and an unconstrained (possibly finite) state-space. A Subjective Expected Utility decision maker in such a setup is characterized by a Tradeoff Consistency assumption that holds unconditionally (see Kobberling and Wakker [13]). The decision maker in the current model, on the other hand, is assumed to satisfy a weakened form of consistency, that holds whenever all acts considered obtain their worst consequence on the same event. Otherwise the decision maker follows an Uncertainty Aversion assumption *á*-la Schmeidler [20]. To compare, the form of consistency assumed here is stronger than the type required for a non-additive expected utility representation, as in Schmeidler [20]. Put together, the consistency axiom of the model and Uncertainty Aversion identify a preference of the decision maker to hedge his or her worst-case scenario, by averaging the worst consequence.

The model is a special case of the neo-additive capacities model of Chateauneuf, Eichberger and Grant [3]. The neo-additive model characterizes preferences that simultaneously overweigh best and worst consequences. Chateauneuf, Eichberger and Grant axiomatized the neo-additive model in a purely subjective framework as the one employed here. The two models are closely related, and the differences originate from the fact that the fundamental feature of a Worst-Case EU decision maker is uncertainty aversion, interpreted as a cautious response to unforeseen events, whereas within the neo-additive model the decision maker may exhibit uncertainty aversion with respect to some events, and uncertainty attraction with respect to others. Accordingly, the Worst-Case EU is characterized by violating Subjective Expected Utility only in an uncertainty averse fashion, while in the neo-additive model these violations depend on the acts involved, specifically on their sharing of events on which best and worst consequences are obtained.

Since the neo-additive model is a special case of the non-additive expected utility model first introduced by Schmeidler [20] (henceforth abbreviated to CEU, for Choquet Expected Utility), so is the Worst-Case EU decision rule. More precisely, the Worst-Case EU rule, unlike the neo-additive rule, is a special case of the CEU rule with a convex non-additive probability[1], thus also of the Maxmin Expected Utility rule of Gilboa and Schmeidler [7] (abbreviated to MEU).

In fact, the Worst-Case Expected Utility model can be formulated as an MEU rule with respect to an ε-contaminated set of priors, where the contaminating set of priors is the entire simplex of probabilities, and ε is endogenous. Such an ε-contamination rule was axiomatized by Kopylov [14][2], however Kopylov's characterization is done in an Anscombe-Aumann framework [2] which includes exogenous probabilities, while the characterization given here is purely subjective. In addition, the view taken here, which is expressed in the different axioms employed, is of the model as a simple, uncertainty averse departure from Subjective Expected Utility.

The Worst-Case EU model is related to models that entertain two associated state spaces. In Jaffray and Wakker [11] and in Mukerji [16] one of the two state spaces assumed is the primitive payoff-relevant state space on which acts are defined, so that each payoff-relevant state is assigned a single consequence. The other state space is an underlying space, on which there is complete probabilistic knowledge through an additive probability measure. Each state in the probabilizable space maps to a set

[1] That is, the non-additive probability ν satisfies, for every pair of events E and F, $\nu(E \cup F) \geq \nu(E) + \nu(F) - \nu(E \cap F)$.

[2] Kopylov also axiomatizes the general case of ε-contamination, see both [14] and [15]. See also Nishimura and Ozaki [18] for an axiomatization of ε-contamination with an exogenous ε.

of payoff-relevant states, thus additive probabilistic evaluations on the first space become non-additive when translated to the second. Jaffray and Wakker show how this structure leads to a preference relation over belief functions, and axiomatize a decision rule over belief functions. Mukerji identifies epistemic foundations which lead to a non-additive probability assessment in this setup, and then discusses the links to a CEU decision rule. Mukerji suggests a cautious mapping of acts from the payoff-relevant space to the underlying, probabilizable space, where each underlying state is assigned the worst consequence obtained on the payoff-relevant states.

Gilboa and Schmeidler [8] show that a CEU decision rule over a state space is equal to an additive representation over a corresponding 'grand' state space. Each state in the 'grand' space represents an event from the original, primitive state space, and acts are extended to the larger state space by assuming their worst consequence on every event. Gilboa and Schmeidler interpret this duality as a sign that the primitive state space is misspecified, and does not contain all actual states in its formulation.

The Worst-Case EU model is a special case of the two-tiered state space. Compared to Jaffray and Wakker [11] and Mukerji [16], the underlying, probabilizable state space contains only one state in addition to the payoff-relevant space – the endogenously derived state, which maps to all payoff-relevant states. In the Gilboa and Schmeidler [8] setup, it is a special case by translating the non-additive probability over the primitive state space to an additive measure over the grand state space, so that only the states which represent either singletons or the entire state space are assigned a non-zero probability. Among these three last papers, only the one by Jaffray and Wakker contains axiomatization. Their axiomatization, however, is given in a setup different from that of the current paper, as they assume a primitive knowledge structure of a two-tiered state-space.

Other, more distantly related models, suppose existence of unforeseen contingencies at the outset, by allowing acts to obtain a *set* of consequences on each state. Such are Nehring [17], Ghirardato [5], Jaffray and Jeleva [10], and Vierø [21], which are consistent with the view that a formulated state may in fact represent a *set* of states which the decision maker cannot tell apart (this is usually referred to as *coarse contingencies*). An even more general approach is taken by Karni and Viero [12], that model a decision maker who gradually becomes aware of new components of the decision problem, consequences and states alike, and responds to those in a reverse Bayesianism manner. By contrast, the model presented in this paper is placed in a standard Savage setup, where the very possibility of unforeseen contingencies is derived from preferences, apparent only in the representation. At the same time, the multi-consequence models mentioned accommodate a richer description of unforeseen elements (i.e. partial knowledge of the consequences obtained on a state) and of the decision maker's attitude towards them.

Lastly I mention that in the context of preferences over lotteries pessimistic departures from expected utility as described here were axiomatized by Gilboa [6] and by Jaffray [9], and departures that may be both pessimistic and optimistic were characterized by Cohen [4] (further elaboration may be found in [3]).

The Worst Case EU representation is aimed to be as simple as possible but still accommodate the question of unforeseen contingencies. Within the representation, the procedure of assigning to the endogenous, 'some unforeseen event occurs' state the worst consequence of each act is but one possibility, which is a simple expression of a cautious conduct. Other modes of extension of acts may be considered, such as assigning to the endogenous state a globally worst consequence, constant for all extended acts. This kind of extension would naturally lead to a standard Subjective Expected Utility rule. However, assigning the endogenous state with the same consequence for each act would mean that this state is, in a sense, independent of all other states, so that what happens on it is independent of the act chosen. By contrast, the choice made here reflects an interpretation that the consequence in case an unforeseen event occurs depends on the act, and the endogenous state does not represent a total catastrophe where all acts collapse to the same worst consequence. Put differently, for each alternative faced, the worst-case scenario *of this alternative* is considered.

The paper is organized as follows. In the next section there is a description of the notation and standard axioms. Section 3 contains an explanation of the Tradeoff Consistency axiom adopted and a formulation of the Uncertainty Aversion assumption of the model. The section concludes with the representation theorem of the model, and an equivalent MEU representation. Proofs are given in Section 4.

2 Notation and basic axioms

Let S denote the set of *states of nature*, endowed with a sigma-algebra of *events*, Σ. Let X be a nonempty set of *consequences*, and $\mathcal{F}$ the set of *acts*, which are taken to be finite-valued mappings from S to X, measurable w.r.t. Σ (i.e., simple acts).

The decision maker's preference relation over acts is denoted by $\succsim$, with $\sim$ and $\succ$ being its symmetric and asymmetric components. With the usual slight abuse of notation, x sometimes denotes the constant act returning x is every state of nature. Similarly, the symbol $\succsim$ is also used to denote a binary relation over X, defined by: $x \succsim y$ whenever the constant act returning x in every state of nature is preferred to the constant act returning y in every state of nature. xEy is the act which assigns the consequence x to the states in E and the consequence y otherwise. An event E is said to be *null* if for all $x, y, z \in X$, $xEz \sim yEz$. For a set of acts $\mathcal{A} \subseteq \mathcal{F}$, E is said to be null on $\mathcal{A}$ if $xEz \sim yEz$ for all $xEz, yEz \in \mathcal{A}$. Otherwise the event is *non-null* (on $\mathcal{A}$).[3]

The first assumption, A0, imposes structural requirements on the set of consequences, thus delimiting the domain of decision problems accommodated by the model.

A0. Structural assumption. S is non-empty, X is a connected topological space, and X^S is endowed with the product topology.

A0 takes a different approach than in Savage [19]. In the Subjective Expected Utility model of Savage the set of consequences is unrestricted, and the set of states is assumed to be rich, whereas here it is the other way around: the set of states is unrestricted (thus may also be finite) and the set of consequences is required to be rich.

Three standard axioms are presented next. In those axioms is it assumed that the relation considered be complete and transitive (A1), continuous (A2), and monotonic (A3).

[3]According to the two nullity definitions, if an event is non-null on a set of acts, then it is non-null. The opposite may not hold.

A1. Weak Order.

(a) For all f and g in X^S, $f \succsim g$ or $g \succsim f$ (completeness).

(b) For all f, g, and h in X^S, if $f \succsim g$ and $g \succsim h$ then $f \succsim h$ (transitivity).

A2. Continuity: The sets $\{f \in X^S \mid f \succ g\}$ and $\{f \in X^S \mid f \prec g\}$ are open for all g in X^S.

A3. Monotonicity: For any two acts f and g, $f \succsim g$ holds whenever $f(s) \succsim g(s)$ for all states s in S.

The fourth assumption implies that the decision problem is not degenerate or deterministic, thus constrains attention to situations that involve some degree of uncertainty.

A4. Essentiality: There exists an event E and consequences $x \succ y$ such that $x \succ xEy \succ y$.

3 Tradeoff consistency and results

A few definitions are first listed, required to present the specific type of tradeoff measurement used in this paper and its corresponding consistency requirement.

Definition 1. For an act f, *min(f)* denotes a worst consequence of f, that is, a consequence $f(s)$ such that $f(t) \succsim f(s)$ for all states t. An act f is said to obtain its worst consequence on an event E if for all $s \in E$, $f(s) \sim \min(f)$.

Definition 2. For a nonempty event T, a *T-worst-consequence set*, denoted by W_T, is the set of all acts which obtain their worst consequence on T. If acts are said to belong to the same worst-consequence set it means that there exists a nonempty event T such that all acts are in W_T.

An act in a set W_T obtains a worst consequence on T, so that $f(s) \succsim f(t)$ for every state $t \in T$ and every state s. Yet, note that it might be the case that a worst consequence is obtained also on states outside of T. Therefore, for instance, each W_T contains all constant acts.

The next relation defines how tradeoffs are measured in this model.

Definition 3. Define a relation $\sim^*$ over pairs of consequences: Let a, b, c, d be consequences. Then $ab \sim^* cd$ if there exist acts f, g and an event E such that,

$$aEf \sim bEg \quad \text{and} \quad cEf \sim dEg \ , \tag{1}$$

where aEf, bEg, cEf and dEg belong to the same W_T for some nonempty event T, and E is non-null on W_T.

According to the definition, tradeoffs are measured using acts that obtain their worst consequence on the same event. In order for this measurement to be meaningful it must be that it is independent of the choice of event and acts used as 'rulers'. This requirement is expressed in the next axiom.

A5. Worst-Consequence Tradeoff Consistency (WC-TC).
Suppose four consequences, a, b, c and d, four acts, f, g, f' and g', and two events, E and F. Then,

$$aEf \sim bEg, \ \ cEf \sim dEg, \ \ aFf' \sim bFg' \ \Rightarrow \ cFf' \sim dFg' \tag{2}$$

whenever aEf, bEg, cEf and dEg belong to W_T for some nonempty event T with E non-null on W_T, and aFf', bFg', cFf' and dFg' belong to W_D for some nonempty event D with F non-null on W_D.

WC-TC is the only point where Worst-Case EU deviates from Subjective Expected Utility. In a purely subjective setup such as assumed here a Subjective Expected Utility decision maker is characterized by maintaining tradeoff consistency for every set of acts involved (along with a few standard axioms, as assumed here; See Kobberling and Wakker [13]). WC-TC weakens the tradeoff consistency assumption needed for Subjective Expected Utility by requiring that there exists an event on which all acts obtain their worst consequence (one event per every foursome in the axiom).

On the other hand, WC-TC is also the only point where Worst-Case EU differs from the CEU model with a convex non-additive probability, given a setup such as here (see [1] and [13]). Such a CEU decision rule is characterized by a tradeoff consistency condition that holds under suitable comonotonicity restrictions.[4] These comonotonicity restrictions imply the worst-case restrictions that appear in the WC-TC axiom, thus WC-TC is stronger than the tradeoff consistency condition required for a CEU model.

Having a consistent way to measure tradeoffs between consequences allows one to identify, for a pair of consequences x and z, a consequence y which is half-way between x and z. This is a consequence y such that the tradeoff between x and y is equivalent to the tradeoff between y and z, namely
$xy \sim^* yz$. State-by-state half-mixtures of acts can then be obtained employing half-way tradeoff mixtures of consequences: for two acts f and g, the act which is half-way between f and g is the act which obtains, in every state s, the half-way consequence between $f(s)$ and $g(s)$. That is to say, a half-mixture of f and g is an act h such that for every state s, $f(s)h(s) \sim^* h(s)g(s)$.

The last axiom employs half-mixtures of acts to formulate an Uncertainty Aversion axiom *á*-la Schmeidler [20].

A6. Uncertainty Aversion. For any three acts f, g and h, if $f \sim g$, and h satisfies, for all states s, $f(s)h(s) \sim^* h(s)g(s)$, then $h \succsim g$.

To understand the role of this axiom in the model it is first noted that assuming axioms A0-A5, $\succsim$ satisfies independence with respect to tradeoff mixtures among acts which obtain their worst consequence on the same event. That is to say, if f, g and h are acts which obtain their worst consequence on the same event, then f is preferred to g if and only if the half-mixture of f and h is preferred to the half-mixture of g and h.

Uncertainty Aversion states that if f is indifferent to g, then the half-mixture of these acts can only make the decision maker better off. Based on the above, if f and g obtain their worst consequence on the same state then their indifference implies

[4]For a definition of comonotonic acts see Definition 4 below.

that they are also indifferent to their half-mixture. Uncertainty Aversion thus asserts that all violations of this independence occur in the same direction, so that in case the indifference is broken it is in favor of the mixture. It implies that the decision maker prefers to average the worst consequence, thus hedging his or her worst-case scenario.[5]

The main theorem of the paper is now stated. The representation it characterizes is as if the decision maker considers an extended state space composed of the primite state space S and an additional endogenous state denoted $\star$. Each act is extended to the new state space, so that the extension of an act f, denoted $f_\star$, is f on S and $\min(f)$ on the endogenous state $\star$. Extended acts are evaluated according to a Subjective Expected Utility functional, with respect to a subjective additive probability on the extended state space. The representation may be read as adding a state signifying 'some unforeseen event occurs', and cautiously extending acts to this state, following a 'worst-case scenario' assumption.

Theorem 1. *Assume a binary relation $\succsim$ on $\mathcal{F}$, where A0 holds. Then the following two statements are equivalent:*

(i) *$\succsim$ satisfies A1-A6.*

(ii) *There exist a continuous utility function $u : X \to \mathbb{R}$, a state $\star$, and a probability measure P over the sigma-algebra generated by $\Sigma \cup \{\star\}$, such that $\succsim$ is represented, for every act f, by the following functional:*

$$V(f) = \int_{S\cup\{\star\}} u(f_\star)dP \quad , \tag{3}$$

where $f_\star : S \cup \{\star\} \longrightarrow X$ satisfies,

$$f_\star(s) = f(s) \quad for \quad s \in S \ , \ f_\star(\star) = \min(f) \ .$$

Furthermore, the utility function u is unique up to unit and location,[6] and the

[5] An Uncertainty Aversion axiom with similar formulation, which employs half-mixtures defined using tradeoffs, appeared in Alon and Schmeidler [1]. The difference is that the axiom here assumes indifference of the acts f and g whereas in [1] weak preference was assumed. Nonetheless, under the rest of the axioms the two formulations are equivalent, and the version with indifference could have been applied in Alon and Schmeidler as well.

[6] That is, unique up to an increasing linear transformation

probability P is unique, satisfying $0 < P(E) < 1$ for some event $E \in \Sigma$.

As discussed above, the set of axioms employed in the theorem falls between the Subjective Expected Utility axioms and the axioms that characterize a CEU rule with a convex non-additive probability. Accordingly, the representation is a weakening of Subjective Expected Utility, as well as a special case of the convex CEU, or, equivalently, a special case of MEU (see Schmeidler [20] for the equivalence of these two decision rules; all rules are meant on the primitive state space S).

All these decision rules derive a cardinal utility function over consequences, and the difference between them lies in their belief component. While the belief of a Subjective Expected Utility maximizer is additive, that of a CEU maximizer that is also an MEU maximizer is non-additive, and only known to satisfy convexity. A Worst-Case EU maximizer is endowed with an 'almost additive' belief, so that the only violation of additivity it exhibits is $\nu(E) + \nu(E^c) \leq 1$, a convex violation. Moreover, $\nu(E) + \nu(E^c) = 1 - P(\star)$ for every event E, hence the additivity violation, which encapsulates the gap between Worst-Case EU and Subjective Expected Utility, is summarized in one endogenous parameter. This parameter can be interpreted as the subjective probability attached to the possibility that an unforeseen event occurs, expressing a degree of caution on the part of the decision maker.

The representation in (ii) of Theorem 1 is also an intermediate rule between Subjective Expected Utility and the neo-additive model of Chateauneuf, Eichberger and Grant [3]. The neo-additive capacity, which is the belief component of that model, is not (necessarily) convex, thus neo-additive preferences do not in general exhibit uncertainty aversion. Instead they accommodate deviations from additivity that are explained by a degree of caution as well as a degree of optimism.

As stated above, Worst-Case EU is a special case of the MEU model. More precisely it is a special case of the ε-contamination MEU representation where the set of priors is generated by ε-contaminating the prior P with the entire simplex of probabilities over Σ, with $\varepsilon = P(\star)$. The degree of contamination is therefore endogenous. As mentioned in the introduction this kind of ε-contamination MEU rule was axiomatized by Kopylov [14]. Nonetheless, the two models bear differences, both technical and conceptual, as detailed in the Introduction.

The ε-contamination representation that corresponds to the Worst-Case EU rule is stated in the next corollary, with $P(\cdot|S)$ denoting the conditional P-probability given S (over Σ), and $\Delta(\Sigma)$ denoting the set of all probability measures over Σ.

Corollary 2. *(i) and (ii) of Theorem 1 are also equivalent to the following MEU representation of* $\succsim$ *, for every act* f*:*

$$V(f) = \min_{\pi \in \mathcal{C}} \int_S u(f) d\pi$$
$$\mathcal{C} = \{(1 - P(\star))P(\cdot|S) + P(\star)\mu \mid \mu \in \Delta(\Sigma)\}$$

where u *and* P *are the utility and probability from (ii) of Theorem 1, and* $0 < \min_{\pi \in \mathcal{C}} \pi(E) < 1$ *for some event* $E \in \Sigma$*.*

4 Proofs

In order to apply results that derive a CEU representation with a convex non-additive probability, a few definitions are required.

Definition 4. A set of acts is *a comonotonic set* if there are no two acts f and g in the set and states s and t, such that, $f(s) \succ f(t)$ and $g(t) \succ g(s)$. Acts in a comonotonic set are said to be *comonotonic acts.*

Definition 5. Given a comonotonic set of acts $\mathcal{A}$, an event E is said to be comonotonically nonnull on $\mathcal{A}$ if there are consequences x, y and z such that $xEz \succ yEz$ and the set $\mathcal{A} \cup \{xEz, yEz\}$ is comonotonic.

Definition 6. Comonotonic Tradeoff Consistency holds if for any four consequences a, b, c, d, states s, t, and acts f, g, f', g',

$$aEf \sim bEg, \ cEf \sim dEg, \ aFf' \sim bFg' \quad \Rightarrow \quad cFf' \sim dFg' \quad ,$$

whenever the sets of acts $\{aEf, bEg, cEf, dEg\}$ and $\{aFf', bFg', cFf', dFg'\}$ are comonotonic, E is comonotonically nonnull on the first set, and F is comonotonically nonnull on the second set.

Note that WC-TC (A5) implies Comonotonic Tradeoff Consistency. According to Theorem 4 in [1], axioms A1-A4, Comonotonic Tradeoff Consistency and A6 are satisfied if and only if $\succsim$ admits a CEU representation, with a continuous utility function u unique up to unit and location, and a unique, non-additive and convex probability ν, satisfying $0 < \nu(E) < 1$ for some event E.[7]

4.1 Proof of Theorem 1: (i) implies (ii)

Claim 3. *Let D, T be two disjoint events, T being nonempty. If D is non-null on W_T, then for every two consequences $x^* \succ x_*$ and act f, $x^*Df \succ x_*Df$, as long as $x^*Df, x_*Df \in W_T$.*

Proof. Suppose on the contrary that $x^*Df \sim x_*Df$ (the opposite cannot be true on account of Monotonicity). Obviously it also holds that $x^*Df \sim x^*Df$ and $x_*Df \sim x_*Df$. Let E be the event which existence is guaranteed by Essentiality, then both E and E^c are non-null on W_{E^c}. From $x^*Ex_* \sim x^*Ex_*$, Worst-Consequence Tradeoff-Consistency (WC-TC) implies $x^*Ex_* \sim x_*Ex_*$. This may be written as $x_*E^cx^* \sim x_*E^cx_*$, which again by WC-TC delivers $x^* = x^*E^cx^* \sim x_*E^cx_* = x_*$. Contradiction. ■

Conclusion 4. *Under the CEU representation of the relation, Claim 3 implies that for every nonempty, disjoint events D and T,*

(a) If D is non-null on W_T, then $\nu(D \cup F) - \nu(F) > 0$ for every event F disjoint from $D \cup T$ (and in particular $\nu(D) > 0$).

(b) If D is null on W_T, then $\nu(D \cup F) - \nu(F) = 0$ for every event F disjoint from $D \cup T$ (and in particular $\nu(D) = 0$).

According to Essentiality there are at least two non-null states. Suppose first that there are exactly two non-null states and denote them by s_1 and s_2 (disregard all other, null states). According to the non-additive representation of $\succsim$, for all acts $f = xs_1y$,

[7] In [1] the Uncertainty Aversion axiom is formulated with a weak preference, $f \succsim g$, rather than indifference, $f \sim g$. However the proof only relies on indifference, based on a proposition from Schmeidler [20] (see Claim 29 and Lemma 30 in [1]).

$$
\begin{aligned}
V(xs_1y) &= \begin{cases} u(x)\nu(s_1) + u(y)[1-\nu(s_1)] & , \quad x \succsim y \\ u(y)\nu(s_2) + u(x)[1-\nu(s_2)] & , \quad y \succ x \end{cases} \\
&= u(x)\nu(s_1) + u(y)\nu(s_2) + (1-\nu(s_1)-\nu(s_2)) \min\left(u(x), u(y)\right) \quad ,
\end{aligned}
$$

with $0 < \nu(s_1) < 1$ or $0 < \nu(s_2) < 1$ by the condition of the theorem in [1]. Set $P(s_i) = \nu(s_i)$ for $i = 1, 2$ and denote $P(\star) = (1 - \nu(s_1) - \nu(s_2))$. By convexity of ν, $P(\star) \geq 0$, and the representation for the two-states case is established.

Otherwise there are at least three non-null states, and the proof proceeds for this case.

In the sequel, the notation $xEyDz$ is employed, for consequences x, y and z, and for disjoint events E and D. It stands for the act which assigns the consequence x to the states in E, the consequence y to the states in D, and the consequence z otherwise.

Let T, D and F be pairwise-disjoint events, with T nonempty. If D or F are null on W_T then by (b) of Conclusion 4, $\nu(D \cup F) = \nu(D) + \nu(F)$. Otherwise suppose that D and F are non-null on W_T. Let x^*, x_0 and x_* be consequences such that $x^* \succ x_0 \succ x_*$ and denote by E the event which existence is guaranteed by Essentiality.

By continuity of u and connectedness of X, and by the non-additive representation, there exist (close enough) consequences x, y and z, $x^* \succ x \succ y \succ z \succ x_0$, such that: (1) There exist consequences a, b with $z \succ b \succ a$, such that $xEa \sim yEb$ and $yEa \sim zEb$, hence $u(x) - u(y) = u(y) - u(z)$. (2) There exists a consequence α such that $x^* \succ \alpha \succ y$ and $xDyFx_* \sim yD\alpha Fx_*$.

WC-TC implies that also $yDyFx_* \sim zD\alpha Fx_*$, as all acts involved belong to W_T. Translating the indifferences into the CEU representation yields:

$$
\begin{aligned}
u(x)\nu(D) + u(y)(\nu(D \cup F) - \nu(D)) &= u(\alpha)\nu(F) + u(y)(\nu(D \cup F) - \nu(F)) \\
u(y)\nu(D \cup F) &= u(\alpha)\nu(F) + u(z)(\nu(D \cup F) - \nu(F))
\end{aligned}
$$

Subtracting the two equations and substituting the utility equalities from (1) delivers $\nu(D) + \nu(F) = \nu(D \cup F)$. Therefore on each W_T the capacity ν is additive, and $\succsim$ is represented by an expected utility functional with the utility function u

and a unique probability measure P_T over Σ, that satisfies $P_T(T) = 1 - \nu(T^c)$ and $P_T(D) = \nu(D)$ for every D disjoint from T. Moreover, for every nonempty T, T' disjoint from D, $P_T(D) = P_{T'}(D) = \nu(D)$.

For every event $D \subsetneq S$, define $P(D) = \nu(D)$, so $P(D \cup F) = P(D) + P(F)$ for every disjoint events D and F such that $D \cup F \subsetneq S$. Every act in $\mathcal{F}$ obtains a finite number of consequences. Denote each act by $f = [x_1, E_1 \ ; \ldots ; \ x_n, E_n]$, so that f obtains, for $i = 1, \ldots, n$, the consequence x_i on the nonempty event E_i, and $\min(f) = x_n$. Employing this notation, the above yields that $\succsim$ over $\mathcal{F}$ is represented by the functional,

$$V(f) = \sum_{i=1}^{n-1} P(E_i)u(x_i) + (1 - \nu(E_n^c))u(x_n) \ ,$$

or, equivalently,

$$\begin{aligned} V(f) \ &= \ \sum_{i=1}^{n} P(E_i)u(x_i) + \delta_{E_n} u(\min(f)) \\ \text{for } \ \delta_{E_n} \ &= \ 1 - \nu(E_n^c) - P(E_n) = 1 - \nu(E_n^c) - \nu(E_n) \ , \end{aligned}$$

where $\delta_{E_n} \geq 0$ by convexity of ν.

Lastly, consider the act xDy for $x \succ y \succ x_0$. For T, T' disjoint from D, this act belongs both to W_T and to $W_{T'}$. Thus (through certainty equivalents),

$$V(xDy) = P(D)u(x) + P(D^c)u(y) + \delta_T u(y) = P(D)u(x) + P(D^c)u(y) + \delta_{T'} u(y) \ ,$$

from which it follows that $\delta_T = \delta_{T'}$.

Let $\star$ denote a new state, and consider the state space $S \cup \{\star\}$, with the sigma-algebra generated by $\Sigma \cup \{\star\}$. Extend P to the new sigma-algebra by $P(\star) = \delta_T = 1 - \nu(T^c) - \nu(T)$ and $P(S) = 1 - \delta_T = \nu(T^c) + \nu(T)$ (for some, hence for all events T), so P is an additive probability measure over $\Sigma \cup \{\star\}$. Maintaining the notation

$f = [x_1, E_1 \,; \ldots ; \, x_n, E_n]$ of acts, $\succsim$ on $\mathcal{F}$ admits the representation,

$$\begin{aligned} V(f) &= \sum_{i=1}^{n} P(E_i)u(x_i) + P(\star)u(\min(f)) \\ &= \int_{S\cup\{\star\}} u(f_\star)dP \ , \end{aligned}$$

for $f_\star : S \cup \{\star\} \longrightarrow X$ that satisfies,

$$f_\star(s) = f(s) \ \text{ for } \ s \in S \ , \ f_\star(\star) = \min(f) \ .$$

The probability P is unique by uniqueness of ν. Since $P(E) = \nu(E)$ for every $E \subsetneq S$ in Σ, then by Theorem 4 in [1] stated above, $0 < P(E) < 1$ for some event E in Σ.

4.2 Proof of Theorem 1: (ii) implies (i)

Suppose that $\succsim$ is represented as in (ii) of Theorem 1. The representation is a special case of a CEU representation, with a capacity ν defined by $\nu(D) = P(D)$ for every event D satisfying $D \subsetneq S$, where $0 < \nu(E) < 1$ for some event E. Thus, according to Theorem 4 in [1] referred to above (in the beginning of this section), axioms A1-A4 and A6 are satisfied. For A5 observe that over each worst-consequence set W_T the representation identifies with an expected utility representation with the same utility function and an additive probability function q, with $q(E) = P(E)$ for E disjoint from T and $q(T) = P(T) + P(\star)$. Hence tradeoff indifference translates to equivalence of utility differences.

4.3 Proof of Corollary 2

According to Theorem 4 in Alon and Schmeidler [1], the axioms imply a CEU representation with respect to a convex non-additive probability, with a continuous utility that is unique up to unit and location, and an event E with non-additive probability strictly between zero and one. The theorem further states that this representation is equivalent to an MEU representation with the same utility, and a set of prior probabilities (over Σ) which is the core of the non-additive probability.

The non-additive probability ν in the representation (3) satisfies $\nu(D) = P(D)$ and $\nu(D) + \nu(D^c) = 1 - P(\star)$ for every event D such that $D \subsetneq S$. The core of this non-additive probability is precisely those priors which allocate probability $P(D)$ to every event $D \subsetneq S$ and distribute the surplus $P(\star)$ over all events $D \in \Sigma$ in all possible ways. Thus the set of priors in the MEU representation is the probability P, ε-contaminated by the entire simplex.

Essentiality and the maxmin representation imply that for some event $E \in \Sigma$, $0 < \min_{\pi \in \mathcal{C}} \pi(E) < 1$.

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