Derivation of a cardinal utility through a weak tradeoff consistency requirement

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Abstract

An axiomatic model is presented, in which a utility function over consequences, unique up to location and unit, is derived. The axioms apply to a binary relation over purely subjective acts, namely no exogenous probabilities are assumed. The main axiom used is a weak tradeoff consistency condition. The model generalizes the biseparable model of Ghirardato and Marinacci (2001).

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1 Introduction.

Decision models under uncertainty are commonly composed of an abstract set of *states of nature*, an abstract set of *consequences*, and mappings from states to consequences, called *acts*. The decision maker's preferences are often modeled through a binary relation over the set of acts. Preferences over consequences are induced from preferences over constant acts (acts yielding the same consequence in every state of nature).

Given a binary relation over acts, one of the most basic questions is whether there is a utility function that represents the preference induced over consequences. If there is a utility function, then it is further interesting to inquire in what sense, if any, the utility function is unique. Many Decision Theory models derive a utility function that is *cardinal*, namely unique up to location and unit, which has the advantage that utility differences between consequences become comparable. This task is typically easier when

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exogenous probabilities are involved, so that acts map states of nature to *lotteries* over a set of consequences, and more demanding when exogenous probabilities are not assumed.

In the literature, one of the techniques applied to derive cardinal utility functions when no exogenous probabilities are available is the *tradeoffs*, or *standard sequences*, technique. This technique originated in Thomsen [11], further developed in Blaschke and Bol [2] and Debreu [3], and was thoroughly investigated in Krantz, Luce, Suppes and Tversky [8]. The tradeoffs technique yields statements such as 'the tradeoff between consequence x and consequence y is the same as the tradeoff between consequence z and consequence w'. The way to produce such statements is to use particular events and acts as measuring devices. For instance, consider consequences α and β and an event E as your measuring device. If an act yielding a consequence x on E and α otherwise is indifferent to an act yielding a consequence y on the same event E and β otherwise, then α, β and E measure the tradeoff, or the 'preference distance', between x and y. Hence, if an act yielding a consequence z on E and α otherwise is indifferent to an act yielding a consequence w on E and β otherwise, then the tradeoff between z and w equals the tradeoff between x and y. The idea is that these indifferences exhibit that under the same circumstances (i.e., event E and consequences α and β) the decision maker is willing to trade x for y as much as z for w. In a similar manner, standard sequences are sequences (not only pairs), in which consequences are equally spaced in terms of 'preference distance'.

It could be the case that all events and all acts are employed as measuring devices, or that only some events, or some acts, are allowed to play that part. In any case, in order for these tradeoff measurements to be consistent, it should be that any one of these devices yields the same results. Wakker, [12] and [13], and Kobberling and Wakker [6] (abbreviated KW from now on), use tradeoff consistency requirements over different sets of events and acts to obtain different representations: all events and all acts for Subjective Expected Utility, comonotonic acts (see Definition 1 below) for nonadditive expected utility $\acute{a} - la$ Schmeidler [10], and so on. Their work assumes an environment that does not contain exogenous probabilities, and delivers, among other parameters of the representation, a cardinal utility.

This paper examines a tradeoff consistency requirement in which the acts allowed to be used to measure tradeoffs are either bets on, or bets against a state, namely comonotonic acts which obtain one consequence on one state and another consequence on all other states. It is shown that even if tradeoff consistency is restricted to comonotonic bets on or against states, and moreover tradeoffs are measured only over single states (and not over their complements), then a cardinal utility function over consequences may still be elicited.[1] This utility function respects tradeoff measurements in the sense that equal tradeoffs imply equal utility differences. The model does not assume existence of exogenous probabilities, but instead requires that the set of consequences be a connected topological space. A restriction of the model is that the state space is assumed finite.

In addition to the elicitation of a cardinal utility, the axioms presented imply local additive representations over sets of bets on, and sets of bets against states. More specifically, denote by $\alpha s\beta$ a bet on state s, with consequences α and β such that α is preferred to β. The relation over the set of all bets on state s satisfies that $\alpha s\beta$ is preferred to $\gamma s\delta$, if and only if, $u(\alpha) + V_{-s}(\beta) \geq u(\gamma) + V_{-s}(\delta)$, where u is a cardinal utility over consequences and V_{-s} is a continuous function that (like u) represents the relation over constant acts. The same is satisfied for bets against states.[2] These additive representations are local in that each holds within one set of bets on a state, or one set of bets against a state, and they cannot be employed to compare bets contingent on different states, nor to compare bets on and against the same state.

The model presented in this paper generalizes the biseparable model of Ghirardato and Marinacci [4]. The biseparable model, which encompasses many known choice models as special cases (for instance, non-additive expected utility of Schmeidler [10], maxmin expected utility of Gilboa and Schmeidler [5]), imposes relatively weak assumptions yet extracts one additive representation over all bets on events, and in particular a cardinal utility over consequences (a similar representation is given by the binary rank-dependent utility model of Luce [9]). As discussed in KW (Section 5.2), a tradeoff consistency requirement over comonotonic bets on events, supplemented with a few basic conditions, yields a biseparable representation. This paper shows that a weaker consistency assumption, that restricts attention to bets on or against *states*, still suffices to derive a cardinal utility. It thus suggests a simpler condition that can be tested in order to establish the use of a cardinal utility over consequences. At the same time, it should be noted that the tradeoff consistency condition is not sufficient to obtain a global additive representation over bets, even if only bets on and against states are considered.

The result in this paper serves to strengthen the validity of the tradeoff method as a tool to empirically measure utility. The tradeoff method elicits a utility by finding consequences which are equally spaced in terms

[1] A few basic axioms are also assumed.

[2] That is, the relation over acts $\alpha s\beta$ with β preferred to α also admits an additive representation $u(\alpha) + V_{-s}(\beta)$, but this representation is different than the one over bets *on* s.

of tradeoffs, under a suitable assumption of tradeoff consistency (for a detailed description and a discussion of the method see Wakker and Deneffe [14], Kobberling and Wakker [7], and the references therein). Assuming that the axioms suggested in this paper are satisfied, the tradeoff method may be applied to derive a cardinal utility over consequences by measuring tradeoffs over single states, using bets on or against states. The representation result introduced in this paper thus extends the domain of preferences to which the tradeoff method applies. Not only is it not required to assume an expected utility representation in order to apply the method, but it is even possible to measure a utility when the relation over binary acts does not conform to the biseparable model (see Example 4).

The paper is organized as follows. The next section presents notation and basic axioms. Section 3 presents the tradeoff consistency axiom used and the result. Section 4 contains an example for a relation which obtains a representation as in the theorem, yet is not a biseparable relation. Section 5 discusses the problem of infinite values and a possible generalization. Finally, all proofs appear in Section 6.

2 Notation and basic axioms.

S	a set of *states of nature*, with typical elements $s, t, \ldots$.
$\Sigma = 2^S$	an algebra of all *events* over S, with typical elements $E, F, \ldots$.
X	a nonempty set of *consequences*, with typical elements $x, y, \ldots$.
$X^S = \{f : S \to X\}$	the set of *acts*: mappings from states to consequences, with typical elements $f, g, \ldots$.
$\succsim$	the decision maker's preference relation: a binary relation over the set of acts X^S. As usual, $\sim$ and $\succ$ denote its symmetric and asymmetric components.
xEy	a notation for the act which assigns the consequence x to the states in E and the consequence y otherwise. $x\{s\}y$ is sometimes abbreviated to xsy (more generally, s is sometimes used instead of $\{s\}$).
$\overline{x}$	a constant act assigning consequence x to every state of nature.

With the usual slight abuse of notation, the symbol $\succsim$ is also used to denote a binary relation on X, defined by: $x \succsim y$ if $\overline{x} \succsim \overline{y}$. A state s is said to be *null* if for all $x, y, z \in X$, $xsz \sim ysz$. For a set of acts $\mathcal{A}$, s is said to be null on $\mathcal{A}$ if $xsz \sim ysz$ whenever $xsz, ysz \in \mathcal{A}$. Otherwise the state is *non-null* (on $\mathcal{A}$). Similarly, an event E is null (on $\mathcal{A}$) if for all $x, y, z \in X$, $xEz \sim yEz$ (whenever $xEz, yEz \in \mathcal{A}$).

The first assumption, A0, restricts the domain of decision problems accommodated by the model. It poses structural requirements on the set of states, as well as on the set of consequences.

A0. Structural assumption:

(a) S is finite.

(b) X is a connected topological space, and X^S is endowed with the product topology.

Three basic assumptions are presented first. Discussion of the axioms is delayed until after their statement.

A1. Weak Order:

(a) For all f and g in X^S, $f \succsim g$ or $g \succsim f$ (completeness).

(b) For all f, g, and h in X^S, if $f \succsim g$ and $g \succsim h$ then $f \succsim h$ (transitivity).

A2. Continuity: The sets $\{f \in X^S \mid f \succ g\}$ and $\{f \in X^S \mid f \prec g\}$ are open for all g in X^S.

A3. Monotonicity: For any two acts f and g, $f \succsim g$ holds whenever $f(s) \succsim g(s)$ for all states s in S.

Assumptions A1-A3 are standard. The first requires that the preference relation be complete and transitive, the second states that it is continuous, and the third that monotonicity holds. Monotonicity implies that the preference over consequences is state-independent. If the consequence $f(s)$ is preferred to the consequence $g(s)$ in each state s, where these are compared through constant acts, then the composition of f out of these consequences cannot make it worse than g.

In order to state the fourth assumption the notion of a comonotonic set of acts is defined, generalizing the definition of pairwise comonotonicity.

Definition 1. *A set of acts is a **comonotonic set** if there are no two acts f and g in the set and states s and t, such that $f(s) \succ f(t)$ and $g(t) \succ g(s)$. Acts in a comonotonic set are said to be **comonotonic acts**.*

Comonotonic acts induce essentially the same ranking of states according to the desirability of their consequences (or, more accurately, the same ranking up to indifferences). Given any numeration of the states, say $\pi : S \to \{1, ..., |S|\}$, the set $\{ \ f \in X^S \mid f(\pi(1)) \succsim \ldots \succsim f(\pi(|S|) \ \}$ is comonotonic. It is a largest-by-inclusion comonotonic set of acts.

A4. Consistent Essentiality:

(a) For all $s \in S$,

$$asx \succ bsx \quad \Rightarrow \quad csy \succ dsy \ \text{ for all } c \succ d,$$

whenever the set $\{asx, bsx, csy, dsy\}$ is comonotonic.

$$xsa \succ xsb \quad \Rightarrow \quad ysc \succ ysd \ \text{ for all } c \succ d,$$

whenever the set $\{xsa, xsb, ysc, ysd\}$ is comonotonic.

(b) There exist distinct states s', s'' and consequences x, y such that $xs'y \succ \bar{y}$ and $\bar{x} \succ ys''x$.

Part (a) of Consistent Essentiality implies that if a state s is non-null on a set of bets on, or a set of bets against s, then it should influence the decision for every pair of acts from this set. The same holds true for essentiality of the complement. Part (b) of the axiom guarantees that there are at least two non-null states. Moreover, applying Monotonicity, $ys''x \succsim xs'y$, therefore part (b) also yields that for some set of bets on, and some set of bets against states, both the state and its complement are non-null.

Consistent Essentiality and Monotonicity can be replaced by a simpler axiom of Dominance: $f \succ g$ whenever $f(s) \succsim g(s)$ for all states s, with strict preference for at least one state. Though simpler, Dominance is a stronger condition as it excludes the possibility of null states. Therefore the less restrictive axioms of Consistent Essentiality and Monotonicity were chosen here.

3 Tradeoff consistency and results.

In order to present the tradeoff measurement used in this paper and its corresponding consistency requirement, two definitions are introduced.

Definition 2. *An act of the form $\alpha s\beta$ is called a* ***simple binary act****. For a state s, a comonotonic set of acts of the form $\alpha s\beta$, either $\{\alpha s\beta \mid \alpha \succsim \beta\}$ or $\{\alpha s\beta \mid \beta \succsim \alpha\}$, is termed a* ***simple binary comonotonic set****.*

Whenever the notation $\mathcal{C}_s$ is used in the sequel to denote a simple binary comonotonic set of acts, it is to be understood that it denotes such a set of the form $\alpha s\beta$.

Definition 3. *Given a comonotonic set of acts $\mathcal{A}$, an event E is said to be* ***comonotonically non-null on*** $\mathcal{A}$*, if there are consequences x, y, z such that $xEz \succ yEz$, and $\mathcal{A} \cup \{xEz, yEz\}$ is comonotonic. Otherwise, E is* ***comonotonically null on*** $\mathcal{A}$*.*

For instance, $\{s\}$ is comonotonically non-null on $\{asx\}$ if it is non-null on the simple binary comonotonic set of acts containing asx. Similarly, if $\mathcal{C}_s$ is a simple binary comonotonic set of acts and $\mathcal{A} \subseteq \mathcal{C}_s$ is any subset of it, then $\{s\}$ is comonotonically non-null on $\mathcal{A}$ whenever it is non-null on $\mathcal{C}_s$.

The relation which determines how tradeoffs are measured under the model is defined next.

Definition 4. *Define a relation $\sim^*$ over pairs of consequences: for consequences a, b, c, d,*
$< a; b > \sim^* < c; d >$ *if there exist consequences x, y and a state s such that,*

$$asx \sim bsy \quad \text{and} \quad csx \sim dsy \tag{1}$$

with all four acts comonotonic and $\{s\}$ comonotonically non-null on the set of four acts.

To get intuition for the relation $\sim^*$, assume that $y \succ x$. In such a case, having $asx \sim bsy$ implies that a is preferred to b in the precise amount that makes the act asx indifferent to the act bsy. In other words, gaining a instead of b on s exactly compensates for the advantage of y over x outside state s. If the same is true for consequences c and d then it is concluded that the decision maker is willing to trade a for b as much as c for d, namely that the tradeoff between a and b is the same as the tradeoff between c and d.

An important point to note is that the relation $\sim^*$ measures the tradeoff between consequences only over single states, and only through bets on one particular state, or bets against one particular state. By contrast, for consequences a, b, c, d, state s and acts f, g, indifferences $asf \sim bsg$ and $csf \sim dsg$ do not in general imply similar equivalence of the tradeoffs $< a; b >$ and $< c; d >$. Moreover, tradeoffs are not even allowed to be measured over complements of single states, so that $xsa \sim ysb$ and $xsc \sim ysd$ still do not imply that the tradeoffs $< a; b >$ and $< c; d >$ are equivalent. The measurement allowed is therefore very cautious. It suggests that indifference relationships which include any violation of comonotonicity, or employ acts which are not simple binary ones, may involve considerations other than the mere tradeoff between consequences.

The definition of $\sim^*$ would be meaningful only if the measurements employed in its definition are independent of the choice of state and consequences outside this state. This is precisely the role of the following axiom.

A5. Simple Binary Comonotonic Tradeoff Consistency (S-BCTC): For any eight consequences $a, b, c, d,\ x, y, z, w$, and states s and t,

$$asx \sim bsy, \quad csx \sim dsy, \quad atz \sim btw \;\Rightarrow\; ctz \sim dtw \tag{2}$$

whenever the sets of acts $\{\ asx, bsy, csx, dsy\ \}$ and $\{\ atz, btw, ctz, dtw\ \}$ are comonotonic, $\{s\}$ is comonotonically non-null on the first set and $\{t\}$ is comonotonically non-null on the second set.

As noted above, only measurement over single states, which employs simple binary comonotonic acts, is involved in the axiom. KW (section 5.2) suggest the following stronger consistency assumption, that applies to all events.

Binary Comonotonic Tradeoff Consistency: For any eight consequences $a, b, c, d,\ x, y, z, w$, and *events* E and F,

$$aEx \sim bEy, \quad cEx \sim dEy, \quad aFz \sim bFw \;\Rightarrow\; cFz \sim dFw$$

whenever the sets of acts $\{\ aEx, bEy, cEx, dEy\ \}$ and $\{\ aFz, bFw, cFz, dFw\ \}$ are comonotonic, E is comonotonically non-null on the first set and F is comonotonically non-null on the second set.

KW explain that Binary Comonotonic Tradeoff Consistency, together with a few basic axioms, yields the biseparable representation of Ghirardato and Marinacci [4]. By contrast, the weakened version of tradeoff consistency introduced in this paper does not deliver a global additive representation over simple binary acts, but only local additive representations, as is further elaborated in the presentation of Theorem 1 below.

Theorem 1, presented next, is the main result of this paper. The theorem is broken into four parts. Parts one through three are the sufficiency parts and consist of implications of axioms A1 through A5. The fourth part provides necessity.

The first part of Theorem 1 states that axioms A1 through A5 imply existence of a cardinal utility over consequences, which respects tradeoff indifferences in the sense that indifference of tradeoffs is translated to equivalence of utility differences.

Theorem 1, Part 1 (Sufficiency): Cardinal utility.
Assume a binary relation $\succsim$ on X^S, where X and S satisfy the structural assumptions, A0(a) and A0(b). Then (i) below implies (ii).

(i) $\succsim$ satisfies A1-A5.

(ii) There exists a continuous function $u : X \longrightarrow \mathbb{R}$, unique up to location and unit [3], such that for all $x, y \in X$, $x \succsim y \iff u(x) \geq u(y)$, and $< a; b > \sim^* < c; d >$ implies $u(a) - u(b) = u(c) - u(d)$.

The second part of Theorem 1 states that axioms A1 through A5 are sufficient to obtain local additive representations over simple binary comonotonic sets. Focusing on the more interesting case when both s and its complement are non-null on a simple binary comonotonic set (otherwise the result is trivial), $\succsim$ on this set admits an additive representation $\alpha s \beta \mapsto u(\alpha) + V_{-s}(\beta)$. That is, the additive value function operating on the single state consequence is the utility function from (ii), whereas the second additive value function, operating on the complement's consequence, is general. The only known attributes of V_{-s} are that it is continuous and that it represents the relation over consequences. Nothing more can be said about the link between functions V_{-s} for different states s or about the link between local additive representations on different simple binary comonotonic sets. It is thus impossible to use these additive representations to compare acts that belong to different simple binary comonotonic sets.

As mentioned above, in the special case where the relation also satisfies Binary Comonotonic Tradeoff Consistency, a biseparable representation results. According to the biseparable representation, if $\alpha E \beta$ denotes a bet on event E (with $\alpha \succsim \beta$), then the relation over bets on events admits the additive representation $\alpha E \beta \mapsto u(\alpha)\rho(E) + u(\beta)(1 - \rho(E))$, with u a cardinal utility function and ρ a non-additive probability. The biseparable model therefore implies one global additive representation over all bets on events, in contrast to the model suggested here, which delivers only *local* additive representations (and only on bets on and against states).

Theorem 1, Part 2 (Sufficiency): Local additive representations on simple binary comonotonic sets.

[3] In other words, unique up to a positive linear transformation.

Assume a binary relation $\succsim$ on X^S, where X and S satisfy the structural assumptions, A0(a) and A0(b). Then (i) above also implies:

(iii) Suppose that $\mathcal{C}_s$ is a simple binary comonotonic set of acts of the form $\alpha s\beta$, on which both s and $\{s\}^c$ are non-null. Then given a function u as in (ii), there exists a function
$V_{-s} : X \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$, unique up to location[4], such that for all $asx, bsy \in \mathcal{C}_s$,

$$asx \succsim bsy \iff u(a) + V_{-s}(x) \geq u(b) + V_{-s}(y) \ . \tag{3}$$

Moreover, the function V_{-s} is continuous where it is finite and represents $\succsim$ on X.

If only s ($\{s\}^c$) is non-null on $\mathcal{C}_s$ then for all $asx, bsy \in \mathcal{C}_s$, $asx \succsim bsy$ if and only if
$u(a) \geq u(b)$ ($u(x) \geq u(y)$).

The third component of the theorem establishes that the axioms imply a general representation of $\succsim$ over all acts, and that general representation identifies with the utility function over constant acts. Some additional attributes of the representation, necessary for the axioms to be satisfied, are stated. One of these attributes requires a definition.

Definition 5. *A function $J : X^S \longrightarrow \mathbb{R}$ is* self-monotonic *if $J(f) \geq J(g)$ whenever*
$J(\overline{f(s)}) \geq J(\overline{g(s)})$ for all states $s \in S$.

Theorem 1, Part 3 (Sufficiency): A general representation.
Assume a binary relation $\succsim$ on X^S, where X and S satisfy the structural assumptions, A0(a) and A0(b). Then (i) above also implies:

(iv) Given a function u as in (ii), there exists a unique continuous and self-monotonic function $J : X^S \longrightarrow \mathbb{R}$, such that for all $f, g \in X^S$, $f \succsim g \iff J(f) \geq J(g)$, and $J(\bar{x}) = u(x)$ for all $x \in X$.

Furthermore, there are distinct states s' and s'' and consequences x and y such that
$u(x) > J(ys''x)$ and $J(xs'y) > u(y)$.

[4]That is, unique up to an additive constant.

Finally the fourth part of the theorem maintains that statements (ii), (iii) and (iv) are necessary for axioms A1-A5 to be satisfied.

Theorem 1, Part 4 (Necessity).
Assume a binary relation $\succsim$ on X^S, where X and S satisfy the structural assumptions, A0(a) and A0(b). Then (ii),(iii) and (iv) together imply (i).

The next corollary is a supplement to the main theorem, and added since it may prove useful in case the theorem is applied.

Corollary 1. *Assume a binary relation $\succsim$ on X^S, where X and S satisfy the structural assumptions, A0(a) and A0(b). Suppose further that the relation satisfies axioms A1-A5. Then there exists $\delta > 0$ such that if $u(a) - u(b) = u(b) - u(c)$, and $|u(a) - u(b)| < \delta$, then $< a; b > \sim^* < b; c >$.*

4 Example.

The following example depicts a relation that satisfies axioms A1 through A5, hence admits the representation and satisfies the conditions in (ii),(iii) and (iv) of Theorem 1. Nonetheless, the relation in the example is not a biseparable one.

Let $S = \{1, 2, \ldots, n\}$ $(n \geq 3)$ and $X = \mathbb{R}_+$. Suppose that $\succsim$ is represented by the following functional $(0 < \varepsilon < 1)$:

$$J(f) = \varepsilon \left(\frac{1}{2}\sqrt{f(s_1)} + \frac{1}{2}\sqrt{f(s_n)} \right) + (1-\varepsilon) \left(\frac{1}{n-2} \sum_{i=2}^{n-1} f(s_i) \right),$$

$$\text{for any act } f \text{ and an ordering } f(s_1) \succsim f(s_2) \succsim \ldots \succsim f(s_n)\ .$$

First observe that this preference relation satisfies all the axioms listed above. It obviously defines a weak order over $(\mathbb{R}_+)^{\{1,2,\ldots,n\}}$, and, being a continuous, monotonic functional over $\mathbb{R}_+$, the implied preference relation satisfies Monotonicity and Continuity. Moreover, for any two acts f and g, $f \succ g$ whenever $f(s) \succsim g(s)$ in all states s and $f(s) \succ g(s)$ for at least one state s. Thus Consistent Essentiality follows, and all states are non-null on any simple binary comonotonic set. Last, assume that for eight consequences a, b, c, d and x, y, v, w, and two states s and t, $asx \sim bsy$, $csx \sim dsy$ and $atv \sim btw$. The first two indifferences imply

$$\frac{\varepsilon}{2}(\sqrt{a} - \sqrt{b}) = \frac{\varepsilon}{2}(\sqrt{y} - \sqrt{x}) + (1-\varepsilon)(y - x) = \frac{\varepsilon}{2}(\sqrt{c} - \sqrt{d})\ \ ,$$

while the third one renders

$$\frac{\varepsilon}{2}(\sqrt{a}-\sqrt{b}) = \frac{\varepsilon}{2}(\sqrt{w}-\sqrt{v}) + (1-\varepsilon)(w-v) \ .$$

The indifference $ctv \sim dtw$ is thus implied, and delivers axiom S-BCTC (A5). Still, this preference relation is *not* a biseparable relation. Biseparable preference relations maintain tradeoff consistency across *events*. Therefore, a biseparable preference would satisfy, for instance, that if $asx \sim bsy$, $csx \sim dsy$ and $vta \sim wtb$, then also $vtc \sim wtd$, whenever the sets $\{asx, bsy, csx, dsy\}$ and $\{vta, wtb, vtc, wtd\}$ are comonotonic. This, however, is not the case here, since $\sqrt{a}-\sqrt{b} = \sqrt{c}-\sqrt{d}$ does not imply $\frac{\varepsilon}{2}(\sqrt{a}-\sqrt{b}) + (1-\varepsilon)(a-b) = \frac{\varepsilon}{2}(\sqrt{c}-\sqrt{d}) + (1-\varepsilon)(c-d)$. [5]

5 Comments.

5.1 Infinite values.

Note that the representation (u, V_{-s}) in (3) may obtain a value of $\pm\infty$. Still, an infinite value may only be obtained on *extreme acts*, as defined below.

Definition 6. *A consequence y is termed* minimal *if $x \succsim y$ for all $x \in X$, and* maximal *if $y \succsim x$ for all $x \in X$. A consequence which is either minimal or maximal is called* extreme. *Correspondingly, a* minimal act *is one which obtains a minimal consequence in every state, whereas a* maximal act *obtains a maximal consequence in every state. An act is said to be* extreme *if it is either minimal or maximal.*

If (u, V_{-s}) represents $\succsim$ on $\{\alpha s\beta \mid \alpha \succsim \beta\}$, then V_{-s} may obtain a value of $+\infty$ on a maximal consequence, thus the representation on a maximal act may equal $+\infty$. Similarly, if (u, V_{-s}) represents $\succsim$ on $\{\alpha s\beta \mid \beta \succsim \alpha\}$, then V_{-s} may obtain a value of $-\infty$ on a minimal consequence, consequently the representation may obtain $-\infty$ on a minimal act. Nevertheless, in any case the utility u is bounded.

The following example shows that an infinite value may be obtained under the conditions of the model (The example is based on Example 3.8 from Wakker [13]). Let $X = [0, 1]$ and $S = \{1, 2, 3\}$. Consider a relation $\succsim$ that over simple binary comonotonic sets of the form $\{\alpha s\beta \mid \alpha \succsim \beta\}$ admits the additive representation $\alpha + \beta$, and over simple binary comonotonic sets

[5] The preference relation defined by J satisfies also tradeoff consistency across complements, though that attribute is not required by the axioms: if $xsa \sim ysb$, $xsc \sim ysd$ and $vta \sim wtb$, then also $vtc \sim wtd$.

$\{\alpha s\beta \mid \beta \succsim \alpha\}$ admits the additive representation $\alpha + \ln\beta$. The resulting relation over constant acts is represented by $u(\alpha) = \alpha$, and satisfies that $< a; b > \sim^* < c; d > \Rightarrow\ a - b = c - d$. Indifference curves on the entirety of $[0,1]^3$ may be completed in any monotonic and continuous manner (for the remaining acts (x_1, x_2, x_3) where the x_i's all differ). The obtained relation satisfies the conditions of Theorem 1.

Denote $V_{-s}(\beta) = \ln\beta$. The additive representation $u(\alpha) + V_{-s}(\beta)$ over simple binary comonotonic sets $\{\alpha s\beta \mid \beta \succsim \alpha\}$ returns the value $-\infty$ on the constant act 0. To see that the consequence 0 must be assigned a value of $-\infty$ by V_{-s}, consider the sequence of consequences $\beta_k = e^{-k}$. For this sequence, $V_{-s}(\beta_k) - V_{-s}(\beta_{k+1}) = V_{-s}(\beta_{k+1}) - V_{-s}(\beta_{k+2})$, therefore V_{-s} must be unbounded from below.

5.2 A generalization of the tradeoff consistency axiom.

Let s be some state. Denote by $\mathcal{A}_s^M$ the set of all acts which obtain their best consequence on s (that is, acts f for which $f(s) \succsim f(t)$ for all states $t \in S$), and by $\mathcal{A}_s^m$ the set of all acts which obtain their worst consequence on s. Consider the following tradeoff consistency axiom, which is a strengthening of Simple Binary Comonotonic Tradeoff Consistency (S-BCTC):[6]

Extreme Consequence Tradeoff Consistency: For any four consequences a, b, c, d, four acts f, g, f', g', and states s and t,

$$asf \sim bsg,\quad csf \sim dsg,\quad atf' \sim btg' \ \Rightarrow\ ctf' \sim dtg'$$

whenever all acts $\{asf, bsg, csf, dsg\}$ belong to $\mathcal{A}_s^M$ or all belong to $\mathcal{A}_s^m$, all acts $\{atf', btg', ctf', dtg'\}$ belong to $\mathcal{A}_t^M$ or all belong to $\mathcal{A}_t^m$, and correspondingly $\{s\}$ is non-null on $\mathcal{A}_s^M$ or on $\mathcal{A}_s^m$, and $\{t\}$ is non-null on $\mathcal{A}_t^M$ or on $\mathcal{A}_t^m$.

Extreme Consequence Tradeoff Consistency strengthens S-BCTC as it imposes consistency when acts are not necessarily binary. Although tradeoffs are still measured over single states, now the acts employed in the measurement are any acts which obtain their extreme consequence – either best for all acts or worst for all acts – on these single states. An interesting feature, satisfied under this axiom and axioms A0 through A4, may be identified: by connectedness of X, weak order and continuity, for each act f and state s there exists a *conditional certainty equivalent* $x \in X$ that satisfies $f \sim fsx$. Extreme Consequence Tradeoff Consistency implies that this conditional certainty equivalent is independent of the specific consequence obtained on s, as

[6] I thank an anonymous referee for pointing out to me this generalization and its interest.

long as this consequence is extreme. More accurately, $asf \sim asx$ if and only if $bsf \sim bsx$, whenever both asf and bsf belong to $\mathcal{A}_s^M$ or both belong to $\mathcal{A}_s^m$. This independence is implied by applying the axiom to indifference relationships $asf \sim asf$, $bsf \sim bsf$, and $asf \sim asx$, which result in $bsf \sim bsx$, if $asf, bsf \in \mathcal{A}_s^M$ or $asf, bsf \in \mathcal{A}_s^m$.[7] Further implications of this axiom remain a matter for future research.

6 Proofs.

6.1 Proof of Theorem 1: Sufficiency.

Two standard observations are listed first. These observations will be used in the sequel, sometimes without explicit reference.

Observation 1. *Consistent Essentiality and Monotonicity imply that there are two consequences* $x^*, x_* \in X$ *such that* $x^* \succ x_*$.

Observation 2. *Weak order, Continuity and Monotonicity imply that each act* f *has a certainty equivalent, i.e., a constant act* $\overline{x}$ *such that* $f \sim \overline{x}$.

The proof that (i) of Theorem 1 implies (ii), (iii) and (iv) is conducted in two logical steps. First, an additive representation is shown to hold on simple binary comonotonic sets. Second, a utility is derived, satisfying the conditions in (ii), and yielding the specific representation in (iii) on simple binary comonotonic sets. A general representation as in (iv) follows by employing certainty equivalents.

6.1.1 Additive representation on simple binary comonotonic sets.

This subsection contains a proof that $\succsim$ on simple binary comonotonic sets is represented by an additive functional, every component of which is a representation of $\succsim$ over constant acts.

Let s be some state and $\mathcal{C}_s$ a corresponding simple binary comonotonic set of acts. Suppose that both s and its complement $\{s\}^c$ are non-null on $\mathcal{C}_s$.

Lemma 2. *The binary relation* $\succsim$ *on* $\mathcal{C}_s$ *is a continuous weak order, satisfying Monotonicity.*

Proof. All attributes on $\mathcal{C}_s$ follow from their counterparts on X^S, assumed in A1, A2 and A3. ■

[7]The other acts involved must, by the other axioms, belong to the same sets. Some nullity conditions are omitted for the sake of brevity.

The proof that $\succsim$ on $\mathcal{C}_s$ admits an additive representation makes use of Theorem 3.2 and Remark 3.7 from Wakker [13]. In order to apply these results two additional attributes are required.

Definition 7. *The binary relation $\succsim$ satisfies the **hexagon condition** on $\mathcal{C}_s$ if, for all acts $\nu s\alpha,\ \mu s\beta,\ \nu s\beta,\ \mu s\gamma,\ \xi s\alpha,\ \nu s\beta,\ \xi s\beta,\ \nu s\gamma$ in $\mathcal{C}_s$,*

$$\nu s\alpha \sim \mu s\beta,\quad \nu s\beta \sim \mu s\gamma,\quad \xi s\alpha \sim \nu s\beta \quad \Rightarrow \quad \xi s\beta \sim \nu s\gamma\ .$$

Definition 8. *The binary relation $\succsim$ satisfies **strong monotonicity** on $\mathcal{C}_s$ if, for all consequences α, β and acts $\alpha sx, \beta sx, ys\alpha$ and $ys\beta$ in $\mathcal{C}_s$,*

$$\begin{array}{lcl} \alpha \succsim \beta & \Leftrightarrow & \alpha sx \succsim \beta sx\ , \text{and} \\ \alpha \succsim \beta & \Leftrightarrow & ys\alpha \succsim ys\beta\ . \end{array}$$

Claim 3. *The binary relation $\succsim$ on $\mathcal{C}_s$ satisfies strong monotonicity and the hexagon condition.*

Proof. Strong monotonicity follows from Monotonicity, and the assumption that both s and $\{s\}^c$ are non-null on $\mathcal{C}_s$ together with part (a) of Consistent Essentiality. The hexagon condition is implied by assigning in (2) both states to s, and $a = d = \nu,\ b = \mu,\ c = \xi,$
$x = \alpha,\ y = v = \beta,\ w = \gamma.$ ■

Lemma 4. *(A combination of Theorem 3.2 and Remark 3.7 from Wakker [13], applied to $\mathcal{C}_s$) The following two statements are equivalent:*

(a) The binary relation $\succsim$ on $\mathcal{C}_s$ is a continuous weak order, satisfying strong monotonicity and the hexagon condition.

(b) There are functions $V_s, V_{-s} : X \to \mathbb{R} \cup \{-\infty, +\infty\}$, such that the binary relation $\succsim$ on $\mathcal{C}_s$ is represented by the additive functional:

$$(V_s, V_{-s})(\alpha s\beta) = V_s(\alpha) + V_{-s}(\beta)\ , \tag{4}$$

with V_s and V_{-s} that are continuous where they are finite.

In case $\mathcal{C}_s = \{\alpha s\beta \mid \alpha \succsim \beta\}$ then V_s and V_{-s} are finite, except possibly V_s at minimal consequences and V_{-s} at maximal consequences. In case $\mathcal{C}_s = \{\alpha s\beta \mid \beta \succsim \alpha\}$, then V_s and V_{-s} are finite, except possibly V_s at maximal consequences and V_{-s} at minimal consequences.

The representation (V_s, V_{-s}) is unique up to locations and joint unit on $\mathcal{C}_s \setminus \{$extreme acts$\}$.[8]

Remark 5. *If values (finite or infinite) at extreme consequences are chosen in the following manner, then the conditions of Lemma 4 are still satisfied (see section 4.2 in Wakker [13]): In case $\mathcal{C}_s = \{\alpha s \beta \mid \alpha \succsim \beta\}$, if an extreme consequence exists, set V_s(minimal consequence) $= \inf\{V_s(x) \mid x \in X \setminus \{$minimal consequences$\}\}$ for any minimal consequence, and V_{-s}(maximal consequence) $= \sup\{V_{-s}(x) \mid x \in X \setminus \{$maximal consequences$\}\}$ for any maximal consequence. In case $\mathcal{C}_s = \{\alpha s \beta \mid \beta \succsim \alpha\}$, if an extreme consequence exists, set V_s(maximal consequence) $= \sup\{V_s(x) \mid x \in X \setminus \{$maximal consequences$\}\}$ for any maximal consequence, and V_{-s}(minimal consequence) $= \inf\{V_{-s}(x) \mid x \in X \setminus \{$minimal consequences$\}\}$ for any minimal consequence.*

In the remainder of the proof, whenever there is a reference to an additive representation over a simple binary comonotonic set, it is assumed that values on extreme consequences are set as in the remark.

Conclusion 6. *Each of V_s, V_{-s} represents $\succsim$ on X.*

Proof. Suppose that $\mathcal{C}_s = \{\alpha s \beta \mid \alpha \succsim \beta\}$. By strong monotonicity and the non-nullity assumptions, applying the additive representation,

$$\alpha \succsim \beta \ \Leftrightarrow \ V_s(\alpha) + V_{-s}(\beta) \geq V_s(\beta) + V_{-s}(\beta) \ .$$

In case β is not maximal, $V_{-s}(\beta)$ is finite and $V_s(\alpha) + V_{-s}(\beta) \geq V_s(\beta) + V_{-s}(\beta) \Leftrightarrow V_s(\alpha) \geq V_s(\beta)$. If β is maximal then $\alpha \succsim \beta \Leftrightarrow \alpha \sim \beta$, and by Monotonicity and Consistent Essentiality there exists $x \in X$ such that $\beta \succ x$. Strong Monotonicity yields $\alpha \sim \beta \Leftrightarrow \alpha s x \sim \beta s x$, yielding $V_s(\alpha) + V_{-s}(x) = V_s(\beta) + V_{-s}(x) \Leftrightarrow V_s(\alpha) = V_s(\beta)$, as $V_{-s}(x)$ is finite.

Similarly,

$$\alpha \succsim \beta \ \Leftrightarrow \ V_s(\alpha) + V_{-s}(\alpha) \geq V_s(\alpha) + V_{-s}(\beta) \ .$$

In case α is not minimal, $V_s(\alpha)$ is finite and the above holds if and only if $V_{-s}(\alpha) \geq V_{-s}(\beta)$. If α is minimal then so is β, and there exists $y \in X$ such that $y \succ \alpha$. Strong Monotonicity again yields $\alpha \sim \beta \Leftrightarrow y s \alpha \sim y s \beta$, implying $V_s(y) + V_{-s}(\alpha) = V_s(y) + V_{-s}(\beta)$. As $V_s(y)$ is finite, it follows that $V_{-s}(\alpha) = V_{-s}(\beta)$, and V_{-s} too represents $\succsim$ on X.

The proof for the case $\mathcal{C}_s = \{\alpha s \beta \mid \beta \succsim \alpha\}$ is completely analogous. ■

[8] That is, if (W_s, W_{-s}) is another additive representation as above, then $W_s = \sigma u + \mu^1$, $W_{-s} = \sigma V_{-s} + \mu^2$, $\sigma > 0$, on non-extreme acts.

6.1.2 Completion of the proof that (i) implies (ii), (iii) and (iv).

Lemma 7. *(A partial version of Lemma VI.8.2 from Wakker [12]):*
Let $\mathcal{C}_s$ and $\mathcal{C}_t$ be two simple binary comonotonic sets of acts. Assume that s and its complement $\{s\}^c$ are non-null on $\mathcal{C}_s$, and that t and $\{t\}^c$ are non-null on $\mathcal{C}_t$. Let (V_s, V_{-s}) and (V_t, V_{-t}) be the implied continuous cardinal additive representations of $\succsim$ on $\mathcal{C}_s$ and $\mathcal{C}_t$, respectively (according to Lemma 4). Then on $X \setminus \{\text{extreme consequences}\}$, $V_t = \sigma V_s + \tau,\ \sigma > 0$.

Proof. By applying Lemma VI.8.2 of Wakker [12] for V_s, V_t (in his notation, for $V_l^{\pi} = V_t$,
$V_k^{\pi'} = V_s$) on $X \setminus \{\text{extreme consequences}\}$ (in case extreme consequences exist, Proposition VI.9.5 from Wakker [12] guarantees that the required topological conditions on $X \setminus \{\text{extreme consequences}\}$ are satisfied, and the Lemma may be applied there). ■

Let s' be a state, and x, y consequences, as characterized in part (b) of Consistent Essentiality. By assumption, $xs'y \succ \overline{y}$, which by Monotonicity implies $x \succ y$. Part (b) of Consistent Essentiality also states that for some other state s'', $\overline{x} \succ ys''x$. Applying Monotonicity once more, $ys''x \succsim xs'y$, hence $\overline{x} \succ xs'y \succ \overline{y}$, implying that both s' and $\{s'\}^c$ are non-null on $\{\alpha s'\beta \mid \alpha \succsim \beta\}$. According to the proof in the previous subsection there exists a continuous cardinal additive representation $(V_{s'}, V_{-s'})$ of $\succsim$ on the simple binary comonotonic set in question, where $V_{s'}, V_{-s'}$ represent $\succsim$ on X. Let $u(x) = V_{s'}(x)$ for all $x \in X \setminus \{\text{minimal consequences}\}$. If a minimal consequence exists, set $u(\text{minimal consequence}) = \inf\{u(x) \mid x \in X \setminus \{\text{minimal consequences}\}\}$. According to Lemma 4, Conclusion 6 and Remark 5, u represents $\succsim$ on X, it is finite except possibly at minimal consequences, and continuous where it is finite.

Let s'' be a state as characterized in part (b) of Consistent Essentiality. Applying Monotonicity as is done above, both s'' and $\{s''\}^c$ are non-null on $\{\alpha s''\beta \mid \beta \succsim \alpha\}$, thus according to the previous subsection there exists a continuous cardinal additive representation $(V_{s''}, V_{-s''})$ of $\succsim$ on this simple binary comonotonic set, where $V_{s''}, V_{-s''}$ represent $\succsim$ on X, and $V_{s''}$ is finite except possibly at maximal consequences. By Lemma 7 $V_{s''}$ is a positive linear transformation of u on non-extreme consequences. If a minimal consequence exists, then $V_{s''}$, being finite there, is bounded from below. Hence u is bounded from below and by its definition on minimal consequences must be finite on those as well. The definition of u on minimal consequences extends its continuity to X.

It follows that $\succsim$ on X is represented by a continuous utility function, $u : X \to \mathbb{R}$. Uniqueness of u and the fact that it respects tradeoff indifferences is proved in the following lemma.

Lemma 8. *If* $< a; b > \sim^* < c; d >$, *then* $u(a) - u(b) = u(c) - u(d)$. u *is unique up to a positive linear transformation.*

Proof. If $< a; b > \sim^* < c; d >$, then there exist consequences x, y and a state t such that $atx \sim bty$ and $ctx \sim dty$, with $\{atx, bty, ctx, dty\}$ comonotonic, and $\{t\}$ comonotonically non-null on this set of acts. If $\{t\}^c$ is comonotonically *null* on $\{atx, bty, ctx, dty\}$, then having $bty \sim btx$ delivers $atx \sim bty \;\Leftrightarrow\; a \sim b$, and similarly $ctx \sim dty \;\Leftrightarrow\; c \sim d$. The equality $u(a) - u(b) = u(c) - u(d)$ is trivially implied.

Otherwise, assume that $\{t\}^c$ is comonotonically non-null on $\{atx, bty, ctx, dty\}$. By the above subsection there exists a continuous additive representation (V_t, V_{-t}) for $\succsim$ on the simple binary comonotonic set containing $\{atx, bty, ctx, dty\}$. Translating the indifference relationships with this representation obtains $V_t(a)+V_{-t}(x) = V_t(b)+V_{-t}(y)$ and $V_t(c)+V_{-t}(x) = V_t(d)+V_{-t}(y)$. If V_t or V_{-t} obtain a value of $\pm\infty$ on one of these consequences, then the comonotonicity restrictions and Consistent Essentiality imply that $a \sim b$ and $c \sim d$, immediately implying the required result. Else, if all values involved are finite, $V_t(a) - V_t(b) = V_t(c) - V_t(d)$. According to Lemma 7, $V_t = \sigma V_{s'} + \tau,\ \sigma > 0$, on non-extreme consequences, thus, as all values are finite and recalling the definition on extreme consequences, also $u(a) - u(b) = u(c) - u(d)$.

For uniqueness of u, let $\hat{u}$ be some other representation of $\succsim$ on X which satisfies that if $< a; b > \sim^* < c; d >$ then $\hat{u}(a) - \hat{u}(b) = \hat{u}(c) - \hat{u}(d)$. Both u and $\hat{u}$ represent $\succsim$ on X, therefore $\hat{u} = \varphi \circ u$, with φ a continuous and strictly increasing transformation. Non-triviality of $\succsim$, connectedness of X and continuity of u imply that $u(X) = V_{s'}(x)$ is a non-degenerate interval. For the same reasons $V_{-s'}(X)$ is a non-degenerate interval. Let ξ be an internal point of $u(X)$. There exists an interval R small enough around ξ such that, for all $u(a), u(c)$ and $u(b) = [u(a)+u(c)]/2$ in R, there are $x, y \in X$ for which $as'x, bs'y, bs'x, cs'y$ are comonotonic, with $a \succsim x,\ b \succsim y,\ b \succsim x$ and $c \succsim y$, and

$$u(a)-u(b) = V_{s'}(a)-V_{s'}(b) = V_{-s'}(y)-V_{-s'}(x) = V_{s'}(b)-V_{s'}(c) = u(b)-u(c).$$

That is, $as'x \sim bs'y$ and $bs'x \sim cs'y$ with $\{as'x, bs'y, bs'x, cs'y\}$ comonotonic, which is precisely the definition of $< a; b > \sim^* < b; c >$. By the assumption on $\hat{u}$, $\hat{u}(a) - \hat{u}(b) = \hat{u}(b) - \hat{u}(c)$ as well, implying that for all $\alpha, \gamma \in R$, φ satisfies $\varphi((\alpha + \gamma)/2) = [\varphi(\alpha) + \varphi(\gamma)]/2$. By Theorem 1 of section 2.1.4 of Aczel [1], φ must be a positive linear transformation. ■

Let s be some state and $\mathcal{C}_s$ a corresponding simple binary comonotonic set of acts. If both s and its complement $\{s\}^c$ are non-null on $\mathcal{C}_s$, then according to Lemma 4, Lemma 7 and the definition of u, $\succsim$ on $\mathcal{C}_s$ admits a representation as in (4) (by choosing for V_s the same location and unit as those of u). If only s is non-null on $\mathcal{C}_s$ then for every asx, bsy in $\mathcal{C}_s$, $asx \succsim bsy$ if and only if $a \succsim b$. Since u represents $\succsim$ on X, it follows that $asx \succsim bsy$ if and only if $u(a) \geq u(b)$. If only $\{s\}^c$ is non-null on $\mathcal{C}_s$ then $asx \succsim bsy$ if and only if $x \succsim y$, which is true if and only if $u(x) \geq u(y)$.

Given u, a unique continuous representation J may be defined using certainty equivalents: for every act f, set $J(f) = u(x)$ for $x \in X$ such that $f \sim \bar{x}$ (such a consequence x exists by Observation 2). The resulting functional J represents $\succsim$ on X^S, is unique given a specific u, and continuous due to continuity of $\succsim$. Self-monotonicity of J follows from Monotonicity (A3). Part (b) of Consistent Essentiality (A4) implies that there are states s' and s'' and consequences x and y such that $J(x) = u(x) > J(ys''x)$ and $J(xs'y) > J(y) = u(y)$.

6.2 Proof of Theorem 1: Necessity.

Let (ii), (iii) and (iv) of Theorem 1 hold. By (iv), $\succsim$ satisfies Weak Order (A1), Continuity (A2) and Monotonicity (A3). By existence of states s', s'' and consequences x, y as detailed in (iv), part (b) of A4 holds. Part (a) of A4 is implied by (iii).

For S-BCTC (A5), let consequences a, b, c, d and x, y, v, w, and states s and t, be such that

$$asx \sim bsy, \quad csx \sim dsy, \quad atv \sim btw \tag{5}$$

where the sets $\{\ asx, bsy, csx, dsy\ \}$ and $\{\ atv, btw, ctv, dtw\ \}$ are comonotonic, $\{s\}$ is comonotonically non-null on the first set and $\{t\}$ is comonotonically non-null on the second set. The first two indifference relations imply, by definition, $< a; b > \sim^* < c; d >$, and thus, according to (ii), $u(a) - u(b) = u(c) - u(d)$. If $\{t\}^c$ is comonotonically null on $\{\ atv, btw, ctv, dtw\ \}$, then $atv \sim btw$ implies $u(a) = u(b)$, which renders $u(a) - u(b) = u(c) - u(d) = 0$ and thus $c \sim d$. From nullity of $\{t\}^c$ and Monotonicity it follows that $ctv \sim ctw \sim dtw$. Otherwise, by Lemma 7, $V_t = \sigma u + \tau$, $\sigma > 0$, hence $V_t(a) - V_t(b) = V_t(c) - V_t(d)$. The indifference $atv \sim btw$ implies $V_t(a) - V_t(b) = V_{-t}(w) - V_{-t}(v)$, thus also $V_t(c) - V_t(d) = V_{-t}(w) - V_{-t}(v)$, yielding $ctv \sim dtw$.

6.3 Proof of Corollary 1.

Let a, b, c be consequences satisfying $u(a) - u(b) = u(b) - u(c)$, and assume w.l.o.g. $u(a) > u(b) > u(c)$, so equivalently $a \succ b \succ c$. Suppose further that

$u(a) - u(b) < \delta$. Let s' be a state satisfying $\bar{x} \succ xs'y \succ \bar{y}$ whenever $x \succ y$ (exists by Essentiality (A4) and Monotonicity), and $(u, V_{-s'})$ the corresponding continuous cardinal additive representation of $\succsim$ on $\{\alpha s'\beta \mid \alpha \succsim \beta\}$. Similarly, let s'' be a state satisfying $\bar{x} \succ ys''x \succ \bar{y}$ whenever $x \succ y$, and $(u, V_{-s''})$ the corresponding continuous cardinal additive representation of $\succsim$ on $\{\alpha s''\beta \mid \beta \succsim \alpha\}$.

Employing the above representations, tt suffices to show that there are consequences x, y that satisfy either

$$u(a) - u(b) = V_{-s'}(y) - V_{-s'}(x), \quad a \succ b \succ c \succsim y \succ x \tag{6}$$

or

$$u(a) - u(b) = V_{-s''}(y) - V_{-s''}(x), \quad y \succ x \succsim a \succ b \succ c\ . \tag{7}$$

Note that according to the conditions on the consequences and Lemma 4, $V_{-s'}$ and $V_{-s''}$ in the above equations must obtain finite values.

Let $z_1 > z_2 > z_3 > z_4$ be interior consequences (there are x^*, x_* for which $x^* \succ z_1 \succ z_2 \succ z_3 \succ z_4 \succ x_*$). If $u(a) - u(c) < \min_{i=1,2,3} (u(z_i) - u(z_{i+1}))$, then either $a \prec z_2$, or $a \succsim z_2$ and $c \succ z_3$. In the first case, letting $u(a) - u(b) < V_{-s''}(z_1) - V_{-s''}(z_2)$ guarantees existence of x, y as required by (7). In the second case, $u(a) - u(b) < V_{-s'}(z_3) - V_{-s'}(z_4)$ guarantees that there are consequences x, y such that (6) is satisfied. Hence, choosing

$$\delta = \min\{\frac{1}{2} \min_{i=1,2,3} (u(z_i) - u(z_{i+1})), V_{-s'}(z_3) - V_{-s'}(z_4), V_{-s''}(z_1) - V_{-s''}(z_2)\}$$

yields the desired result. That is, if $u(a) - u(b) < \delta$ then there are x, y satisfying either (6) or (7), implying $< a; b > \sim^* < b; c >$. ■

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