A comment on the axiomatics of the Maxmin Expected Utility model*

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Abstract

Maxmin Expected Utility was first axiomatized by Gilboa and Schmeidler [6] in an Anscombe-Aumann setup [2] which includes exogenous probabilities. The model was later axiomatized in a purely subjective setup, where no exogenous probabilities are assumed. The purpose of this note is to show that in all these axiomatizations the only assumptions that are needed are the basic ones that are used to extract a cardinal utility function, together with the two typical Maxmin assumptions, Uncertainty Aversion and Certainty Independence, applied only to $0.5 : 0.5$ mixtures. For the purely subjective characterizations this means that assumptions involving an unbounded number of variables can be replaced with assumptions that involve only a finite number thereof.

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1 Introduction

Gilboa and Schmeidler [6] (GS) proposed an axiomatic characterization of a Maxmin Expected Utility (MEU) decision maker, who evaluates alternatives according to their minimum expected utility with respect to a set of prior probabilities. The characterization was obtained in an Anscombe-Aumann setup [2] (as rephrased by Fishburn in [4]), namely in an environment that contains both subjective and objective uncertainty. Later on, Casadesus-Masanell, Klibanoff and Ozdenoren [3], Ghirardato, Maccheroni, Marinacci and Siniscalchi [5] and Alon and Schmeidler [1] (AS) characterized the MEU decision rule in a framework that includes only subjective uncertainty. All the MEU axiomatizations, the original one and the three that followed, begin with a derivation of a cardinal utility over consequences. All of them proceed with employing mixtures in order to obtain the desired representation.

In the original model, placed in the Anscombe-Aumann (AA) framework, the utility extracted is a von-Neumann and Morgenstern [11] utility over lotteries, and the mixtures employed are state-wise lottery mixtures. Two additional axioms that employ those mixtures are imposed. The first is Uncertainty Aversion (due to Schmeidler, [9]), expressing a preference of the decision maker toward hedging. Uncertainty Aversion states that when the decision maker mixes an act g with an act f that is at least as preferred as g, the resulting mixture is still at least as good as g. Uncertainty Aversion departs from Independence, the main axiom characterizing expected utility decision making in an AA framework, by allowing the decision maker to prefer the mixture of f and g to the more preferred act f, a preference that cannot be represented under Independence and Expected Utility. The second GS axiom that employs mixtures is Certainty Independence, which asserts that preference is preserved under mixtures with constant acts (i.e. with lotteries). In both these axioms, mixtures of any proportion can be performed.

The other three papers mentioned above derive a representation in a setup that does not contain objective probabilities, therefore lottery mixtures are not available. Instead, a rich set of consequences is assumed. Each of the three papers utilizes this richness, formulating its own set of assumptions that lead to the elicitation of a cardinal utility over the set of consequences. Mixtures of consequences are then expressed in terms of preference in different manners, where all methods are shown to imply mixtures of utilities once the cardinal utility has been derived. Namely, in each of these methods the $0.5 : 0.5$ preference mixture, for example, of two consequences x and z, is a consequence y that satisfies $u(y) = 0.5u(x) + 0.5u(z)$, u being the cardinal utility derived from preference.

To express mixtures (by any proportion) Casadesus-Masanell, Klibanoff and Ozdenoren [3] employ the notion of a standard sequence, which is a sequence of consequences with a constant preference-distance, as measured by two fixed consequences and a fixed event.[1] Ghirardato, Maccheroni, Marinacci and Siniscalchi [5] employ preference midpoints which are defined based on a mixing event and certainty equivalents, and use these midpoints for a limiting definition of mixtures of any proportion. In AS an MEU representation is derived in two different manners. One characterization relies on preference midpoints as in [5], but only on midpoints, without the limiting definition. The other characterization employs the construct of tradeoffs and the definition of tradeoff consistency as used by Wakker, specifically definitions and results from Kobberling and Wakker [7].[2] In all three papers, the next step after identifying a notion of mixtures is analogous to GS, where Uncertainty Aversion and Certainty Independence are formulated using the paper-specific mixtures, and the representation follows.

It is by now a known fact that in the presence of a proper continuity condition Uncertainty Aversion can be phrased using only $0.5 : 0.5$ mixtures, and some of the papers mentioned employ only such mixtures in the formulation of this condition. By contrast, all the MEU models except for AS assume a Certainty Independence condition that involves mixtures with any possible proportion, or at least any possible rational one. The axiomatization in AS includes versions of Uncertainty Aversion and Certainty Independence that involve only $0.5 : 0.5$ mixtures, however these two conditions are supplemented in AS by an additional assumption, called Certainty Covariance. Certainty Covariance, together with the other two axioms, is shown to imply the general-mixture version of Certainty Independence.

The purpose of this note is to show that Certainty Covariance, the third axiom used in AS, in fact follows from Uncertainty Aversion and Certainty Independence in their $0.5 : 0.5$ formulations. The characterization in AS can therefore be amended to include only the basic axioms that are used to extract a cardinal utility function, together with the two MEU-typical axioms, Uncertainty Aversion and Certainty Independence, applied only to $0.5 : 0.5$ mixtures. Moreover, this result readily follows in all the other models. First, their current Certainty Independence condition can be derived from its simpler $0.5 : 0.5$ version using the same arguments, once a cardinal utility has been derived. Second, in GS,

[1] See Krantz et al [8] for an elaborate discussion of standard sequences.

[2] Tradeoffs serve as a tool for measuring and comparing preference-distances between pairs of consequences. Wakker in several papers employs tradeoffs and different versions of consistency of tradeoff to derive a variety of representations (such as Subjective Expected Utility, Choquet Expected Utility, Prospect Theory) in purely subjective frameworks (see [7] as well as, e.g., [12] and [10]).

where Certainty Independence is also employed for deriving an affine utility over lotteries, its $0.5 : 0.5$ version implies its general, $\alpha : 1-\alpha$ version, if a proper continuity condition is assumed.

Taking the number of variables involved in an axiom as a simple measure of its complexity, weakening the MEU-typical axioms so that they refer only to $0.5 : 0.5$ mixtures yields axioms that are less complex compared to their general-mixture versions in [3] and [5]. The idea of evaluating the complexity of axioms using the number of variables they involve, or even more roughly, distinguishing between axioms that involve a bounded number of variables and those that require an unbounded number, is discussed in detail in AS. If that distinction is accepted, the purely-subjective counterparts of Uncertainty Aversion and Certainty Independence that employ only $0.5 : 0.5$ mixtures, and thus require a fixed number of variables, are considerably less complex than their counterparts with general mixtures, which require an unbounded number of variables (See Subsections 2.2 and 3.2 in AS for a further explanation of these axioms and the number of variables they involve).

A characterization of an MEU decision maker that uses axioms that are as simple and as transparent as possible is desirable, whether a normative or a descriptive role of such a characterization is considered. From a descriptive stance, simpler axioms are easier to test. On the other hand, when considering a normative interpretation, decision makers need to understand the preference rules described in axioms in order to determine whether they agree with them, and the less complex the axioms are, the easier it is to grasp them.

2 Result

In the four papers mentioned above the first step in the extraction of an MEU representation is a derivation of a utility function over consequences, which is cardinally unique and linear w.r.t. the model-specific mixtures considered.[3] GS in the AA framework consider a vNM utility function over lotteries, namely a utility which is linear w.r.t. probability mixtures of lotteries.

In the purely subjective frameworks of the other three papers the set of consequences is a connected topological space, and mixtures are defined in different manners based on the richness of the consequences set. In [3] and in the first development in [1] mixtures are based on the notion of standard sequences, or tradeoffs. In essence, both measure preference distances using two fixed consequences and an event. According to these notions,

[3]Cardinal uniqueness requires an assumption of non degeneracy, which simply implies that trivial cases are avoided.

the preference distance between two pairs of consequences will be the same whenever there are two consequences x, y and an event A such that in each of these pairs, receiving the first consequence instead of the second on A exactly compensates for receiving x instead of y outside of A. In [5] and in the second development in [1] mixtures are defined through certainty equivalents, where the mixing uses a fixed event. In any case, eventually, after a cardinal utility over the set of consequences is derived, all mixtures are shown to reduce to utility mixtures. That is, the $\alpha : 1-\alpha$ mixture of two consequences x and z is another consequence y that satisfies $u(y) = \alpha u(x) + (1-\alpha)u(z)$.

This paper is concerned with the extraction of an MEU representation after a linear utility has been appropriately derived. The first step of deriving the utility in the different frameworks is therefore abstracted away, and attention is restricted to the problem phrased already in utility space. The preference relation we consider is defined over functions from S to a non-degenerate interval $I \subseteq \mathbb{R}$, to be interpreted as the state-wise utility translation of acts from the original models (I is the utilities' image of the set of consequences, which can be open or closed, finite or infinite, on either end). In the background there is an underlying assumption that for every two consequences x and z and every proportion $\alpha \in [0,1]$, there exists a consequence which is both the $\alpha : 1-\alpha$ utility mixture, and the $\alpha : 1-\alpha$ model-specific preference mixture of x and z.

The identification of model-specific and utility mixtures is true for GS, where any two lotteries can be randomized by any $\alpha : 1-\alpha$ probabilities, yielding a lottery that is the desired mixture and has a vNM utility which is the $\alpha : 1-\alpha$ mixture of the vNM utilities of the original lotteries. The equivalence between preference mixtures and utility mixtures also holds for the purely subjective models of Casadesus-Masanell et al [3] and Ghirardato et al [5]. In AS the identity of utility and model-specific mixtures holds for a restricted domain of alternatives, on which the MEU representation is initially derived. The representation is then extended to the entire domain by an attribute termed there Certainty Covariance, which is shown below to be implied by Uncertainty Aversion and Certainty Independence in their $0.5 : 0.5$ versions. Thus for AS as well an MEU representation on the entire domain of acts follows from the proof given here.

It should be noted that in GS, Certainty Independence in its general, $\alpha : 1-\alpha$ form, is used to derive the affine utility function over lotteries, the stage that is omitted here.[4] However, with a proper continuity condition (continuity in the weak topology), the $0.5 : 0.5$ version of Certainty Independence implies its general version, and thus suffices to derive an

[4] I thank an anonymous referee for this comment.

affine utility over lotteries. The proof is identical to that of Claim 1 in the next section. In the other, purely subjective papers, general mixtures are not involved in the first stage of deriving utilities over outcomes.

Formally, let S be a set of states and Σ a sigma algebra of events over S, and let $I \subseteq \mathbb{R}$ be a non-degenerate interval. The set of alternatives is $B_0(I,\Sigma)$, the set of Σ-measurable functions on S which assume finitely many values in I. For $y \in I$ we denote by y^S the constant function which returns y for every state $s \in S$. A binary relation $\succsim$ is supposed on $B_0(I,\Sigma)$. Four basic attributes are first assumed on $\succsim$ (All four are implied by the basic axioms assumed in the four MEU models discussed, without the assumptions of Uncertainty Aversion and Certainty Independence).

Weak Order. For any $f, g \in B_0(I,\Sigma)$, either $f \succsim g$ or $g \succsim f$. For any $f, g, h \in B_0(I,\Sigma)$, if $f \succsim g$ and $g \succsim h$ then $f \succsim h$.

Monotonicity. For any $f, g \in B_0(I,\Sigma)$, $f \succsim g$ whenever $f(s) \geq g(s)$ for every $s \in S$.

It follows trivially that for two constant functions x^S and y^S, $x^S \succsim y^S$ if and only if $x \geq y$.

Continuity. Let $f, g, h \in B_0(I,\Sigma)$ and $f_n \in B_0(I,\Sigma)$ a sequence of functions that state-wise converges to f. If $f_n \succsim g$ for every n, then $f \succsim g$, and if $h \succsim f_n$ for every n, then $h \succsim f$.

In each of the specific setups mentioned above, the AA setup or the purely subjective setup, this form of continuity follows from the specific continuity axiom applied (either Archimedeanity or topological continuity).

Non Degeneracy. There exist $f, g \in B_0(I,\Sigma)$ such that $f \succ g$.

Next, $0.5 : 0.5$ mixtures are employed to formulate the MEU-specific assumptions, Uncertainty Aversion and Certainty Independence. For $f, g \in B_0(I,\Sigma)$, the mixture $\frac{1}{2}f + \frac{1}{2}g$ is an alternative $h \in B_0(I,\Sigma)$ such that for every $s \in S$, $h(s) = \frac{1}{2}f(s) + \frac{1}{2}g(s)$. In all the four MEU papers mentioned, the definition of mixtures through preferences reduces to the definition used here once a utility over consequences is derived.

Uncertainty Aversion (UA). For $f, g \in B_0(I,\Sigma)$, if $f \succsim g$ then $\frac{1}{2}f + \frac{1}{2}g \succsim g$.

Certainty Independence (CI). For $f, g \in B_0(I, \Sigma)$ and a constant function y^S, $f \succsim g$ if and only if $\frac{1}{2}f + \frac{1}{2}y^S \succsim \frac{1}{2}g + \frac{1}{2}y^S$.

Theorem 1. *Let $\succsim$ be a binary relation over $B_0(I, \Sigma)$. Then the following two statements are equivalent:*

(i) $\succsim$ satisfies Weak Order, Monotonicity, Continuity, Non Degeneracy, Uncertainty Aversion and Certainty Independence.

(ii) There exists a unique non-empty, closed and convex set C of additive probability measures on Σ, such that, for all $f, g \in B_0(I, \Sigma)$,

$$f \succsim g \quad \Leftrightarrow \quad \min_{P \in C} \int_S f dP \geq \min_{P \in C} \int_S g dP \ . \tag{1}$$

Remark 1. The theorem could be proved with Uncertainty Aversion and Certainty Independence stated for $\lambda : 1-\lambda$ mixtures for some fixed λ, not necessarily $\lambda = 0.5$. For that, these axioms would need to be formulated so that their conclusions would hold for both $\lambda : 1 - \lambda$ and $1 - \lambda : \lambda$ mixtures, and the proof could then be repeated analogously.[5]

2.1 Proof

It is straightforward to see that (ii) implies the axioms in (i). We prove that the reverse is also true. For that, note first that Weak Order, Continuity and Monotonicity imply that each $f \in B_0(I, \Sigma)$ admits a certainty equivalent, i.e., a constant function x^S such that $f \sim x^S$. Next, applying Uncertainty Aversion consecutively implies that if $f, g \in B_0(I, \Sigma)$ satisfy $f \succsim g$, then $\frac{k}{2^m}f + (1 - \frac{k}{2^m})g \succsim g$, for $k, m \in \mathbb{N}$, $\frac{k}{2^m} \in [0, 1]$. Continuity then yields that the same is true for any $\alpha : 1 - \alpha$ mixture of f and g ($\alpha \in [0, 1]$). That is to say, Uncertainty Aversion holds for any $\alpha : 1 - \alpha$ mixture.

Claim 1. *Let x^S, y^S, w^S be constant acts and $\alpha \in (0, 1)$. Then*

$$x^S \succsim y^S \iff \alpha x^S + (1 - \alpha)w^S \succsim \alpha y^S + (1 - \alpha)w^S \ .$$

[5] I thank an anonymous referee for suggesting this remark.

Proof. First suppose that $x^S \succsim y^S$. By Certainty Independence it follows that

$$\frac{1}{2}x^S + \frac{1}{2}w^S \succsim \frac{1}{2}y^S + \frac{1}{2}w^S \ . \tag{2}$$

By applying Certainty Independence once more, mixing (2) with w^S, it follows that $\frac{1}{4}x^S + \frac{3}{4}w^S \succsim \frac{1}{4}y^S + \frac{3}{4}w^S$. On the other hand, if we mix x^S into (2), followed by $\frac{1}{2}y^S + \frac{1}{2}w^S$, we obtain, $\frac{1}{2}\left(\frac{1}{2}x^S + \frac{1}{2}w^S\right) + \frac{1}{2}x^S \succsim \frac{1}{2}\left(\frac{1}{2}y^S + \frac{1}{2}w^S\right) + \frac{1}{2}x^S \succsim \frac{1}{2}\left(\frac{1}{2}y^S + \frac{1}{2}w^S\right) + \frac{1}{2}y^S$, that is, $\frac{3}{4}x^S + \frac{1}{4}w^S \succsim \frac{3}{4}y^S + \frac{1}{4}w^S$. We can repeat analogous steps to obtain that for every $k, m \in \mathbb{N}$, $\frac{k}{2^m} \in (0,1)$, $\frac{k}{2^m}x^S + (1 - \frac{k}{2^m})w^S \succsim \frac{k}{2^m}y^S + (1 - \frac{k}{2^m})w^S$. Continuity yields that $\alpha x^S + (1-\alpha)w^S \succsim \alpha y^S + (1-\alpha)w^S$.

Suppose that for two constant acts z^S and t^S, $z^S \sim t^S$. Then according to the previous paragraph, $\beta z^S + (1-\beta)w^S \sim \beta t^S + (1-\beta)w^S$, for every $\beta \in (0,1)$. Finally suppose that $x^S \succ y^S$. From the previous paragraph we know that $\alpha x^S + (1-\alpha)w^S \succsim \alpha y^S + (1-\alpha)w^S$, but not necessarily with strict preference. However, repeating the same steps as above it follows that for every $k, m \in \mathbb{N}$, $\frac{k}{2^m} \in (0,1)$, $\frac{k}{2^m}x^S + (1 - \frac{k}{2^m})w^S \succ \frac{k}{2^m}y^S + (1 - \frac{k}{2^m})w^S$. Moreover, if $\alpha x^S + (1-\alpha)w^S \sim \alpha y^S + (1-\alpha)w^S$, let $k, m \in \mathbb{N}$ be such that $\frac{k}{2^m} < \alpha$ and set $\beta = \frac{k}{\alpha 2^m}$, hence by the previous argument, $\beta(\alpha x^S + (1-\alpha)w^S) + (1-\beta)w^S = \frac{k}{2^m}x^S + (1 - \frac{k}{2^m})w^S \sim \beta(\alpha y^S + (1-\alpha)w^S) + (1-\beta)w^S = \frac{k}{2^m}y^S + (1 - \frac{k}{2^m})w^S$. Contradiction. Therefore $\alpha x^S + (1-\alpha)w^S \succ \alpha y^S + (1-\alpha)w^S$. ■

Claim 2. *Let $f \in B_0(I, \Sigma)$, x^S a constant act such that $f \sim x^S$, and $\alpha \in (0,1)$. If $g = \alpha f + (1-\alpha)x^S$, then $g \sim x^S$.*

Proof. The claim is first proved for $\alpha = \frac{1}{2^m}$, that is, when $g = \frac{1}{2^m}f + (1 - \frac{1}{2^m})x^S$. The proof is by induction on m. For $m = 1$ the indifference follows by Certainty Independence. Assume that for $g = \frac{1}{2^m}f + (1 - \frac{1}{2^m})x^S$, $g \sim x$. Let $h = \frac{1}{2^{m+1}}f + (1 - \frac{1}{2^{m+1}})x^S$, then $h = \frac{1}{2}g + \frac{1}{2}x^S$, and $g \sim x$ by the induction assumption. Employing Certainty Independence again implies $h \sim x$.

Now let $\alpha = \frac{k}{2^m}$ ($k \in \{2, \ldots, 2^m - 1\}$), so that $g = \frac{k}{2^m}f + (1 - \frac{k}{2^m})x^S$. By Uncertainty Aversion, $g \succsim x$. We assume $g \succ x$ and derive a contradiction.

Let $g' = \frac{1}{2}g + \frac{1}{2}x^S$, and let y^S be a constant act such that $g \sim y^S$. Employing Certainty Independence, $g' \sim z^S$ for $y > z = \frac{x+y}{2} > x$, so $g \succ g' \succ x^S$. However, there exists an act h such that $h = \alpha g + (1-\alpha)g'$ and also $h = \frac{1}{2^m}f + (1 - \frac{1}{2^m})x^S$ for some $\alpha \in (0,1)$ and some $m \in \mathbb{N}$. By the previous paragraph, $h \sim x^S$. But Uncertainty Aversion requires that $h \succsim g'$, contradiction. Therefore if $g = \frac{k}{2^m}f + (1 - \frac{k}{2^m})x^S$ then $g \sim x^S$. By continuity it follows that the same is true for any $\alpha : 1-\alpha$ combination of f and x^S. ■

Claim 3. *Let $f, g \in B_0(I, \Sigma)$ and x^S and y^S two constant functions. If $\frac{1}{2}f+\frac{1}{2}y^S \sim \frac{1}{2}g+\frac{1}{2}x^S$, then $f \sim x^S$ if and only if $g \sim y^S$.*

Proof. Suppose that $\frac{1}{2}f + \frac{1}{2}y^S \sim \frac{1}{2}g + \frac{1}{2}x^S$ and $f \sim x^S$. By Certainty Independence, $f \sim x^S$ implies $\frac{1}{2}f + \frac{1}{2}y^S \sim \frac{1}{2}x^S + \frac{1}{2}y^S$. Suppose on the contrary that $g \succ y^S$, then again by Certainty Independence, $\frac{1}{2}g + \frac{1}{2}x^S \succ \frac{1}{2}y^S + \frac{1}{2}x^S$. Contradiction. If on the other hand $y^S \succ g$ a contradiction is similarly implied since Certainty Independence now yields that $\frac{1}{2}y^S + \frac{1}{2}x^S \succ \frac{1}{2}g + \frac{1}{2}x^S$. ■

Claim 4. *Let $f \in B_0(I, \Sigma)$, w^S, x^S, y^S constant functions, and $\alpha \in (0,1)$ such that $f \sim x^S$ and $y = \alpha x + (1-\alpha)w$. If $h = \alpha f + (1-\alpha)w^S$, then $h \sim y^S$.*

Proof. Let $g = \alpha f + (1-\alpha)x^S$, then by Claim 2, $g \sim x^S$. In addition, $h = \alpha f + (1-\alpha)w^S = g + y^S - x^S$, that is, $\frac{1}{2}h + \frac{1}{2}x^S = \frac{1}{2}g + \frac{1}{2}y^S$. As $g \sim x^S$, Claim 3 implies that $h \sim y^S$. ■

Suppose $f, g \in B_0(I, \Sigma)$ and let x^S and y^S be constant functions such that $f \sim x^S$ and $g \sim y^S$. Then by Claim 4 for any constant function w^S and any $\alpha \in (0,1)$, $\alpha f + (1-\alpha)w^S \sim \alpha x^S + (1-\alpha)w^S$, and $\alpha g + (1-\alpha)w^S \sim \alpha y^S + (1-\alpha)w^S$. Employing also Claim 1 it follows that:

$$\begin{aligned} f \succsim g \iff x^S \succsim y^S &\iff \alpha x^S + (1-\alpha)w^S \succsim \alpha y^S + (1-\alpha)w^S \\ &\iff \alpha f + (1-\alpha)w^S \succsim \alpha g + (1-\alpha)w^S \ . \end{aligned}$$

Since certainty equivalents exist for all acts it is established that Certainty Independence holds for any $\alpha : 1-\alpha$ mixture. It follows that in all the MEU models mentioned Uncertainty Aversion and Certainty Independence can be weakened to state their implication only with regard to $0.5 : 0.5$ mixtures. In AS, the assumption of Certainty Covariance (A8 in AS) may be dropped, as a result of the above development and the remark below.

Remark 2. In AS, an assumption by the name Certainty Covariance (A8 in AS) is used to extend the MEU representation from a sub-domain of alternatives on which it is initially obtained to the domain of all the alternatives. In the setup and notation of this paper, Certainty Covariance states that if, for $f, g, x^S, y^S \in B_0(I, \Sigma)$, $f - g = x^S - y^S$, then $f \sim x^S$ if and only if $g \sim y^S$. Since $f - g = x^S - y^S$ is equivalent to $\dfrac{f + y^S}{2} = \dfrac{g + x^S}{2}$, Claim 3 implies this attribute and the MEU representation may be extended to the entire domain of alternatives on the basis of the assumptions stated above, specifically Uncertainty Aversion and Certainty Independence for $0.5 : 0.5$ mixtures.

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